# MATH 115 — FINAL EXAM

Department of Mathematics
University of Michigan

April 21, 2003

NAME: <u>Solution Key</u>

INSTRUCTOR: ______________________ SECTION NO: ________________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 11 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. <u>Pay particular attention to the instructions for Problem 1</u>, but read *all* instructions for each individual exercise carefully.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it.
6. If you use graphs or tables to obtain an answer, be certain to provide an explanation and a sketch of the graph to make it clear how you arrived at your solution.
7. Include units in your answers where appropriate.
8. You are allowed two sides of a 3 by 5 card of notes. You may also use your calculator.
9. Please turn off all cell phones.

| PROBLEM | POINTS | SCORE |
|---|---|---|
| 1 | 16 | |
| 2 | 5 | |
| 3 | 6 | |
| 4 | 12 | |
| 5 | 9 | |
| 6 | 8 | |
| 7 | 8 | |
| 8 | 9 | |
| 9 | 9 | |
| 10 | 9 | |
| 11 | 9 | |
| TOTAL | 100 | |

**1. True** or **False**–no explanation necessary. Circle **True** only if the statement is *always* true. You are encouraged to answer these problems only if you are sure of your answer.

Scoring will be:

- 2 points for each correct answer,
- 0 points for not answering, and
- -1 point for each incorrect answer.

**Assume that all functions are continuous and differentiable.**

**(a)** If $f$ is increasing, then $f'$ is increasing. **True** | **False** |

**(b)** If $y = \pi^5$, then $y' = 5\pi^4$. **True** | **False** |

**(c)** If $f'$ is increasing, then the graph of $f$ lies above the graph of any line that is tangent to the curve. | **True** | **False**

**(d)** If $f''(a) = 0$, then $f$ has an inflection point at $x = a$. **True** | **False** |

**(e)** If $f''$ is negative at a critical point, then $f$ has a local maximum at that point. | **True** | **False**

**(f)** If $a \neq b$, then $\int_a^b f(x)\,dx \neq 0$. **True** | **False** |

**(g)** If $f(x) \leq g(x)$ for all $x$ on the interval $[2, 6]$, then $\int_2^6 [g(x) - f(x)]\,dx \geq 0$. | **True** | **False**

**(h)** If $\int_a^b (2f(x) + g(x))\,dx = 5$ and $\int_a^b g(x)\,dx = 2$, then $\int_a^b f(x) = 3$. **True** | **False** |

**2.** (5 points) The temperature, $A$, measured in degrees Fahrenheit, of the water near the surface of a small lake $t$ days after the beginning of fall is described by $A = f(t)$.

Explain the meaning of the statement "$f'(30) = -2$".

**Solution:** *Thirty days after the beginning of fall, the temperature of the water near the surface of the lake is decreasing by about 2 degrees Farenheit per day.*

**3.** (6 points) A continuous, differentiable function $f$ is defined for $x \geq 0$, and satisfies

- $f$ has exactly one critical point,
- $f(0) = 0$ and $f(3) = 2$,
- $f'(1) = 0$, and
- $\lim_{x\to\infty} f(x) = 0$.

Circle each of the following conditions that are possible.

**Solution:**

$\boxed{f \text{ has a local maximum at } x = 1.}$

$f$ has a local minimum at $x = 1$.

$f$ has neither a local maximum or a local minimum at $x = 1$.

$\boxed{f \text{ has a global maximum at } x = 1.}$

$f$ has a global minimum at $x = 1$.

**4.** (12 points) **(a)** Give the limit definition of the derivative of a function $f$ at a point $a$.

**Solution:**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

**(b)** Use the limit definition of the derivative to find $g'(x)$ for the function $g(x) = 2x^2 - 3x$. [Be sure to show all of your work!]

**Solution:**

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{\left(2(x+h)^2 - 3(x+h)\right) - \left(2x^2 - 3x\right)}{h} \\ &= \lim_{h \to 0} \frac{\left(2x^2 + 4xh + 2h^2 - 3x - 3h\right) - \left(2x^2 - 3x\right)}{h} \\ &= \lim_{h \to 0} \frac{4xh - 3h + 2h^2}{h} \\ &= \lim_{h \to 0} 4x - 3 + 2h = 4x - 3 \end{aligned}$$

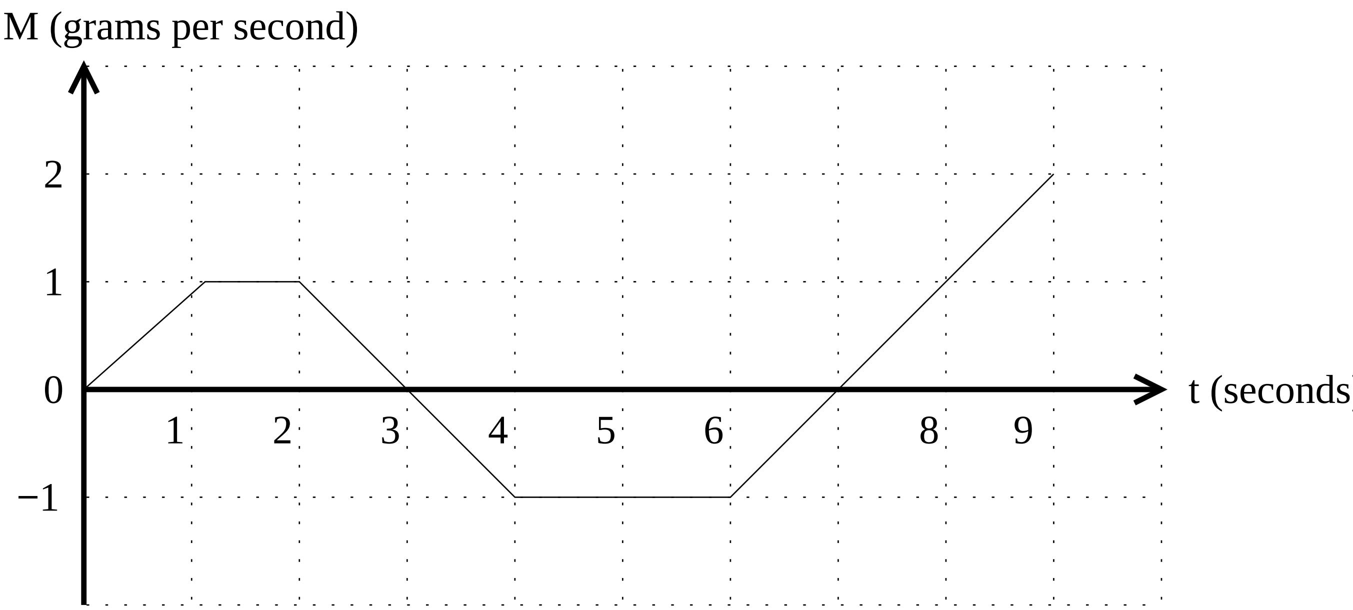
**(c)** Use the Fundamental Theorem of Calculus to find $\int_2^4 (4x - 3)\, dx$. [Note: You must show your work to receive credit.]

**Solution:** *If* $g(x) = 2x^2 - 3x$ *then* $g'(x) = 4x - 3$ *so that*

$$\int_2^4 (4x - 3)\, dx = \int_2^4 g'(x)\, dx = g(4) - g(2) = 20 - 2 = 18$$

*where second equality holds because of the Fundamental Theorem.*

**5.** (9 points) A substance, $B$, is one of several substances involved in a complex chemical reaction. At certain times during this reaction, substance $B$ is produced by the reaction while at other times it plays the role of a reactant and is consumed. Given that enough reactants are present, the rate $M$, of production of substance $B$ is approximated by the function whose graph is given below.




**(a)** Over what interval(s) is the amount of substance $B$ increasing?

**Solution:** *$M = dB/dt$ is positive for $0 < t < 3$ and for $7 < t < 9$ so the amount of substance $B$ present is increasing on those two intervals.*

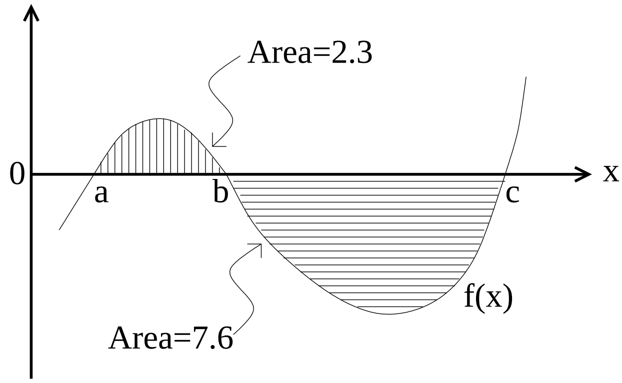
**(b)** At what time during the reaction is the least amount of substance $B$ present? Explain.

**Solution:** *The least amount of substance $B$ is present when $t = 7$. Because, if $B(t)$ denotes the amount of $B$ present $t$ seconds after the beginning of the reaction, then the change in $B$, $\Delta B(t) = B(t) - B(0)$ is equal to the integral of $M$, the rate of change of $B$ over the interval from $0$ to $t$. This shows the amount of $B$ present increases for $0 \leq t \leq 3$ by 2 grams, the area under the graph of $M$ over this interval, so $B(3) = B(0) + 2$ gms. For $3 \leq t \leq 7$, the amount of $B$ present decreases by 3 grams, the area between the graph of $M$ and the $x$-axis over this interval, so $B(7) = B(0) - 1$. And, the amount of $B$ present then increases for $0 \leq t \leq 9$ (up to $B(9) = B(0) + 1$). So, the smallest amount occurs when $t = 7$.*

**(c)** The reaction takes 9 seconds to complete and will not proceed if there is no substance $B$ present. There is a value, $V$, such that if the reaction begins with $V$ or fewer grams of substance $B$, then the reaction will not proceed to completion. Find the value of $V$, and explain your answer.

**Solution:** *The value is $V = 1$. As explained in in part (**b**), the least amount of $B$ is present at $t = 7$ and is $B(0) - 1$ gm, one gram less than at the beginning of the reaction. If there had been less than one gram of $B$ at the beginning, the amount of $B$ would have been exhausted before $t = 7$ so the reaction would not have completed.*

**6.** (8 points) Use the figure below to calculate the numerical values of the definite integrals in parts (a) through (d). You need not show your reasoning.



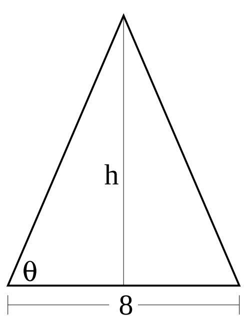

**(a)** $\int_a^b f(x)\,dx =$ 2.3

**(b)** $\int_b^c f(x)\,dx =$ $-7.6$

**(c)** $\int_a^c f(x)\,dx =$ $-5.3$

**(d)** $\int_b^a f(x)\,dx =$ $-2.3$

**7.** (8 points) An isosceles triangle has a base of length 8 meters. If $\theta$ denotes the angle opposite one of the two equal sides, and if $\theta$ is increasing at a constant rate of of 0.1 radians per second, how fast is the area of the triangle increasing when $\theta = \pi/6$?




**Solution:** *If $A(\theta)$ is the area of the triangle with angle $\theta$, then we are asked to find the rate of change of $A$ with respect to time when $\theta = \pi/6$. We are given that $d\theta/dt = .1$ radians per second.*

*Let $h$ denote the height of the triangle. Since the triangle is isosceles, the perpendicular bisector of the base passes through the top vertex so $\tan\theta = h/(\text{half of base}) = h/4$ or $h = 4\tan\theta$. Thus, the area $A$ of the triangle is*

$$A = \frac{1}{2} \textit{base} \cdot \textit{height} = \frac{1}{2}(8) \cdot (4\tan\theta) = 16\tan\theta \quad \textit{square meters.}$$

*From the chain rule and the formula for the derivative of the tangent function, we find $dA/dt = \left(16/\cos^2\theta\right) d\theta/dt$. When $\theta = \pi/6$, $\cos\theta = \sqrt{3}/2$ so at this time,*

$$\frac{dA}{dt} = \frac{16}{(\sqrt{3}/2)^2}\frac{d\theta}{dt} = \frac{64}{3}(.1) = \frac{6.4}{3} \simeq 2.1333 \quad \textit{square meters per second.}$$

**8.** (9 points) The table gives the values of a function obtained from an experiment.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 9.3 | 9.1 | 8.3 | 6.9 | 3.3 | -.6 | -1.7 | -3.5 | -6.7 |

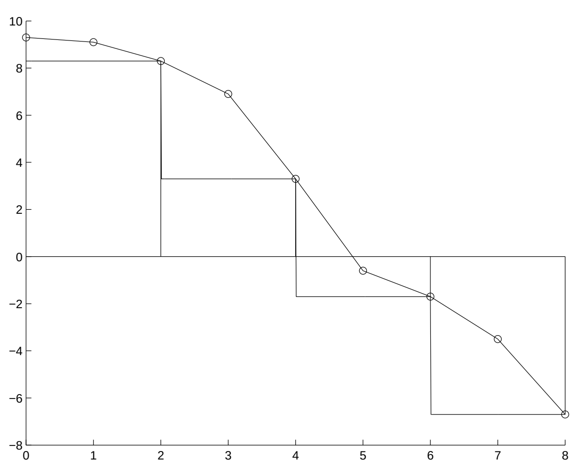
**(a)** Using these values, estimate $\int_0^8 f(x)\,dx$ using 4 subintervals and right hand endpoints.

**Solution:** *The interval has length* 8 *and there are to be* 4 *subintervals so* $\Delta x = 8/4 = 2$. *The estimate using right hand sums is then given by:*

$$\sum_{j=1}^{4} f(x_j)\Delta x = f(2)\cdot 2 + f(4)\cdot 2 + f(6)\cdot 2 + f(8)\cdot 2 = (8.3 + 3.3 + (-1.7) + (-6.7))\cdot 2 = (3.2)\cdot 2 = 6.4.$$

**(b)** If $f$ is known to be a decreasing function, can you determine if your answer in part (a) is an over or underestimate? If so, which is it? If not, why not. Be sure to explain your answer.

**Solution:** *The answer in part* **(a)** *is an underestimate when $f$ is decreasing (see the figure). This is because on each of the subintervals, the value of $f(x)$ is greater than or equal to the value of $f$ at the right hand endpoint, $x_j$ of that subinterval. Therefore, the integral of $f$ over the subinterval is greater than or equal $f(x_j)\Delta x$, the signed area of the rectangle with width $\Delta x$ and height $f(x_j)$. Note that if $f(x_j) < 0$, then this area is negative, and it is a smaller negative number than the integral of $f$ over this subinterval. Adding up these inequalities over all four subintervals shows that the integral is greater than or equal to the sum of the signed areas of the rectangles, i.e. the right hand sum.*



**9.** (9 points) An ecologist is studying the biodiversity of an environment near the top edge of a windswept cliff. One statistic of interest to her is the distribution of biomass throughout the environment. If $x$ measures the horizontal distance from the cliff edge in meters, there is only one species of tussock grass that grows for $1 \leq x \leq 20$. Along the first meter from the cliff's edge, nothing grows; and beyond 20 meters from the cliff, various other plant species thrive.

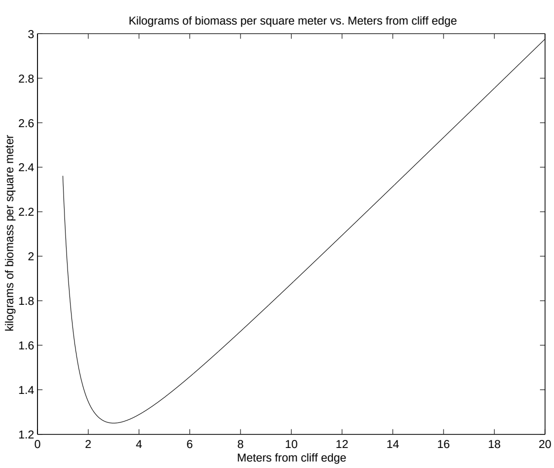
A typical tussock grass plant located $x$ meters from the cliff edge has mass $\frac{2}{9}x^3 + \frac{3}{2}x^2 + 3$ kilograms per plant, and there will be $\frac{1}{2x^2}$ such plants per square meter.

(a) For $1 \leq x \leq 20$, find the distance from the cliff which minimizes the biomass per square meter. Show your work.

**Solution:** *The biomass $B$ per square meter at a distance $x$ meters from the cliff edge is*

$$B(x) = \{\#\ kg/plant\} \times \{\#\ plants/m^2\} = \left(\frac{2}{9}x^3 + \frac{3}{2}x^2 + 3\right) \times \left(\frac{1}{2x^2}\right) = \left(\frac{1}{9}x + \frac{3}{4} + \frac{3}{2x^2}\right)\ kg/m^2$$

*We are asked to find the point $x$ in the interval $1 \leq x \leq 20$ where $B(x)$ takes on its minimum value. Since $B$ is given by a simple formula, we can look at a graph of $B$ (for example, on a calculator) to get an idea of where the point $x$ point might be. A graph of $B$ is shown in the figure below for the relevant values of $x$ and it appears that $B$ has a minimum somewhere near the left end of this range. In particular, $B$ should have one critical point in the interval which is a minimum.*




*To find the point $x$ exactly, calculate: $dB/dx = \frac{1}{9} - \frac{3}{x^3}$. A critical point of $B$ is a solution of the equation $dB/dx = 0$ or $\frac{1}{9} = \frac{3}{x^3}$; that is, where $x^3 = 27$. The only solution in the interval $1 < x < 20$ is $x = 3$. Since the graph shows that $B(x)$ is decreasing for $1 < x < 3$ and increasing for $3 < x < 20$, this must be a global minimum of $B$ on the interval. The minimum value is $B(3) = (5/4)kg/m^2$.*

At a distance of ___*3 meters*___ from the cliff's edge the biomass per square meter is minimized.

(b) What is the maximal biomass per square meter in this region? Explain.

**Solution:** *The maximum of $B(x)$ on the interval must occur either at a critical point or at an endpoint. The graph shows clearly that the maximum occurs at the right hand endpoint, or $x = 20$, so the maximal biomass is $B(20) = 21427/7200 \simeq 2.97597222$ or about $3kg/m^2$.*

*The value of $B$ at $x = 1$ is $B(1) = 85/36 \simeq 2.3611111kg/m^2$, so we see that both the values at the left hand endpoint and the critical point are smaller than the value at the right hand endpoint.*

The maximal biomass per square meter in this region is ___*about $3kg/m^2$*___.

**10.** (9 points) A piece of wire of length 40 cm is cut into two pieces. One piece is made into a circle; the rest is made into a square.

Find the lengths of each piece of wire so that the sum of the areas of the circle and square is a minimum. [Be sure to show *all* of your work and clearly identify your answers.]

**Solution:** *Let $x$ denote the length of wire that is made into the circle, so that $40-x$ is the length of the piece of wire to be shaped into a square. The circumference of the circle is then equal to $x$ so it must have radius $r = x/2\pi$. The square, made from the remaining piece of wire that has length $40-x$, then has side length $(40-x)/4$ so the area of the circle plus the area of the square is equal to*

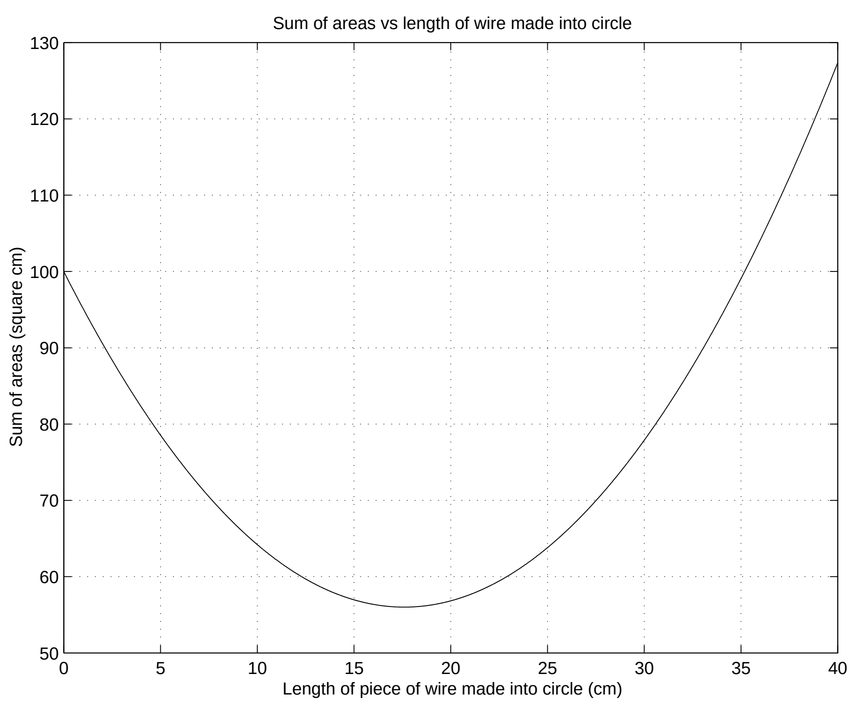
$$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{40-x}{16}\right)^2 = \frac{x^2}{4\pi} + \frac{(40-x)^2}{16}.$$

*We see that the function $A(x)$ is a quadratic polynomial in $x$ and the coefficient of $x^2$ is $(1/4\pi - 1/16)$ which is positive. Therefore, the graph of $A(x)$ is a parabola opening upward so it can have at most one critical point which will be a global minimum. The graph of $A(x)$ is shown in the figure below.*

*To find the critical point of $A$, we could either complete the square to write the quadratic function $A(x)$ in the form $a(x-b)^2+c$, or else find the zero of the derivative $A'(x)$. Here we will use the latter method and compute the solution of the linear equation $A'(x) = 0$. That is,*

$$0 = A'(x) = \frac{x}{2\pi} - \frac{40-x}{8} = x\left(\frac{1}{2\pi} + \frac{1}{8}\right) - 5 = x\left(\frac{8+2\pi}{16\pi}\right) - 5$$

*or* $\boxed{x = 40\pi/(4+\pi) \simeq 17.59603386cm}$. *This value of $x$ is in the interval $0 < x < 40$ so it is the value which gives minimum area. The length of the other piece of wire is* $\boxed{40 - x = 160/(4+\pi) \simeq 22.40396614.}$ *The minimum value of the sum of the areas turns out to be approximately* $56.00991535\ cm^2$



**11.** *(9 points) Let $s(t)$ give the position of an object along a straight line at time $t$ and let $v(t)$ denote its instantaneous velocity at time $t$.*

**(a)** *Give the definition of the* average velocity *of the object over the time interval from $t = a$ to $t = b$.*

**Solution:** The average velocity is

$$\frac{s(b) - s(a)}{b - a}.$$

**(b)** *Give the definition of the* average of the velocity function *over the interval from $t = a$ to $t = b$.*

**Solution:** The definition of the average of the velocity function is:

$$\frac{1}{b-a} \int_a^b v(t)\, dt.$$

**(c)** *Is the* average velocity *of the object over the time interval from $t = a$ to $t = b$ equal to the* average of the velocity function *over this time interval? If so, explain why. If not, explain why not.*

**Solution:** The two quantities are equal because of the fundamental theorem of calculus. That is, from part (b), the average of the velocity function is equal to

$$\frac{1}{b-a} \int_a^b v(t)\, dt = \frac{1}{b-a} \int_a^b \frac{ds(t)}{dt}\, dt.$$

By the fundamental theorem of calculus, the last expression is equal to

$$\frac{1}{b-a}\left(s(b) - s(a)\right)$$

which, by part **a**, is the average velocity.

*Please rewrite your name and section number.*

*NAME:* ______________________________________________

*SECTION NO:* ____________________