

MATH 115 — FIRST MIDTERM EXAM SOLUTIONS

1. (2 points each) Circle “True” or “False” for each of the following problems. Circle “True” only if the statement is *always* true. No explanation is necessary.

(a) $\log\left(\frac{1}{A}\right) = -\log(A)$.

TRUE False

(b) If $f(x) = \pi^5$, then $f'(x) = 5\pi^4$.

True FALSE

(c) The function $y = \frac{a}{b+ce^{-kt}}$ for $k > 0$ and a, b, c constants has a horizontal asymptote of $y = \frac{a}{c}$.

True FALSE

(d) A degree 7 polynomial must have at least 1 real root but can not have more than 7 real roots.

TRUE False

(e) $f'(a)$ is the tangent line of f at the point $(a, f(a))$.

True FALSE

(f) If $f(x) = x^2$, then $f^{-1}(x) = \frac{1}{x^2}$.

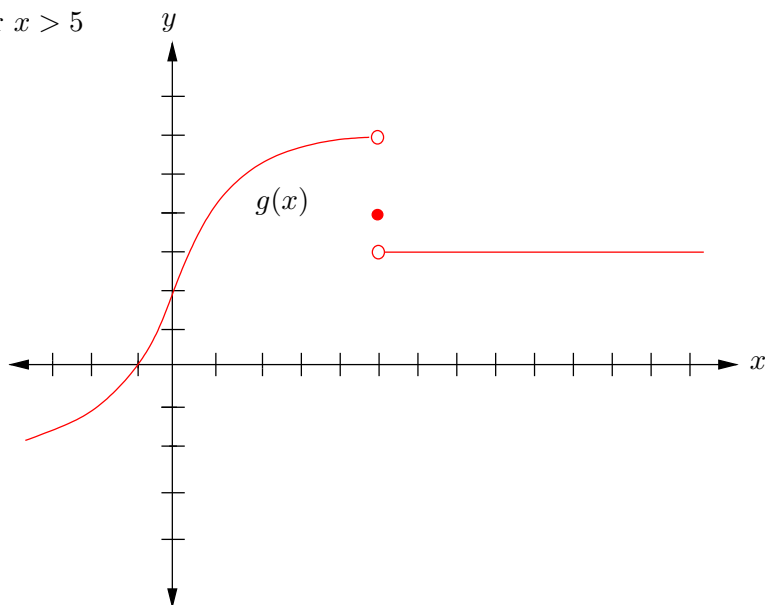
True FALSE

(g) **Note:** This problem was excluded from grading. If $f''(a) = 0$, then the point $(a, f(a))$ is an inflection point of f .

True FALSE

2.(8 points) On the axes below, sketch a graph of a single **function**, g , with **all** of the following properties.

- $g(0) = 2$
- $g'(x) > 0$ for $x < 5$
- $g''(x) > 0$ for $x < 0$
- $g''(x) < 0$ for $0 < x < 5$
- $\lim_{x \rightarrow 5^-} g(x) = 6$ and $\lim_{x \rightarrow 5^+} g(x) = 3$
- $g(5) = 4$
- $g'(x) = 0$ for $x > 5$



3. (1+1+3 points) Upon graduating from the university and landing your first big job, you decide to reward yourself for all the hard work and purchase a brand new sports car. The price of the sports car is \$45,000. The value of the car depreciates at the rate of 37% per year. Comprehensive insurance costs 10% of the car's value each year. For parts (a) and (b) circle the best choice.

(a) The value of the sports car is an exponential function of time.

(b) The cost of the comprehensive insurance is a linear function of V , the value of the car.

(c) Write a function that gives the cost of the comprehensive insurance policy on the car after the t^{th} year.

$$C(t) = 0.10(45,000(0.63)^t).$$

4. (6 points) A circus is planning to visit your hometown. They claim to have a mummy that is 15,000 years old, but the citizens of your town are suspicious. The town council is given a sample of the mummy for a carbon-14 analysis. Your old high school science teacher is able to find that 33% of the mummy's carbon-14 remains. Using the fact that the half-life of carbon-14 is 5,730 years, determine the age of the mummy that the circus is bringing to town. [Show your work!]

We know that since the carbon-14 decays exponentially, that the formula for the amount of carbon-14 left after t years is given by

$$P = P_0 e^{rt}$$

where P_0 is the initial amount and r is the exponential rate of decay.

To find r , we use that the half-life of carbon-14 is 5730 years. So we have

$$\begin{aligned} \frac{1}{2}P_0 &= P_0 e^{5730r} \\ \frac{1}{2} &= e^{5730r} \\ -\ln(2) &= 5730r \\ r &= \frac{-\ln(2)}{5730}. \end{aligned}$$

So now we can find the time elapsed when 33% of the mummy's carbon remains.

$$\begin{aligned} 0.33P_0 &= P_0 e^{\frac{-\ln(2)t}{5730}} \\ \ln(0.33) &= \frac{-\ln(2)t}{5730} \\ t &= 9,164.92 \text{ years} \end{aligned}$$

Therefore, the circus is exaggerating the age of their mummy.

5. (3 points each) The marketing department of Lay's Potato Chips decides to do a study on the number of chips a person craves as a function of the number of chips already eaten by that person. Their function turns out to be the rational function

$$C(x) = \frac{2x^2}{(x-1)^2},$$

where $C(x)$ is the number of additional chips a person who has eaten x chips is still craving to eat.

(a) How is Lay's famous slogan "You can't eat just one" summarized by this equation?

The slogan is given by the fact that 1 is not in the domain of $C(x)$. This shows that it is impossible to eat one Lay's potato chip.

(b) What is the horizontal asymptote of $C(x)$? What does this say about a person with an unlimited supply of chips who can't control his/her cravings?

The horizontal asymptote is at $y = 2$. This says that in the long run a person will always crave 2 more Lay's potato chips. So if one could not control one's cravings, one would eat potato chips forever.

6. (12 points) For this problem f is differentiable everywhere.

(a) Let $g(x) = f(x - 2)$. Describe the graph of $g(x)$ in terms of the graph of $f(x)$.

The graph of g is the graph of f shifted to the right 2 units.

(b) If $f'(1) = 6$, what is $g'(3)$? Don't do any calculations here, use the geometry of the situation from part (a) to arrive at your answer.

Since we have just shifted the graph of f to the right by 2, we must have $g'(3) = f'(1) = 6$. So we are looking at the slope of the tangent line to f at $x = 1$, only shifted along with the graph of g .

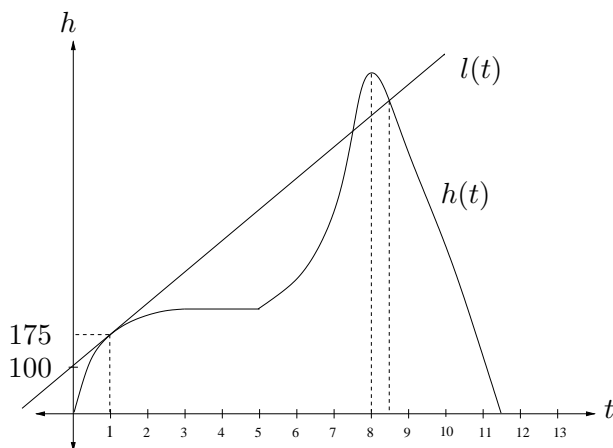
(c) State the limit definition of the derivative for the function f .

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

(d) Let $j(x) = f(x) + 10$. Use the limit definition of the derivative to calculate the derivative of j in terms of the derivative of f .

$$\begin{aligned} j'(x) &= \lim_{h \rightarrow 0} \frac{j(x+h) - j(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(f(x+h) + 10) - (f(x) + 10)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= f'(x). \end{aligned}$$

7. (12 points) The graph below gives a rock climber's height as a function of time as he climbs a small mountain. The height is measured in feet and the time is measured in hours. The line $l(t)$ gives the tangent line to $h(t)$ at time $t = 1$.



(a) For which time(s), if any, is the climber stopped?

The climber is stopped between the 3rd and 5th hour, and then again at the 8th hour.

(b) Does the climber speed up or slow down over the first three hours?

The climber slows down over the first hour, as can be seen by the fact that the graph is concave down on this interval. So he is climbing up at a decreasing rate.

(c) What is the climber's rate of ascent 1 hour into the climb?

The climber's rate of ascent is the derivative of the function $h(t)$, which happens to be the slope of the tangent line $l(t)$. To find the slope of $l(t)$, we use the points $(0, 100)$ and $(1, 175)$.

$$\begin{aligned} \text{slope of } l(t) &= \frac{175 - 100}{1 - 0} \\ &= 75 \text{ ft/hour} \end{aligned}$$

(d) What is the climber's height after 8.5 hours?

Note that the line $l(t)$ intersects $h(t)$ at $t = 8.5$. The equation for $l(t)$ is given by

$$l(t) = 75t + 100$$

. Putting in $t = 8.5$, we have that $h(8.5) = 737.5$ feet.

(e) If the maximum height the climber reaches is 800 feet, what is his average rate of ascent over the last 3.5 hours of his trip (i.e., for $8 < t < 11.5$)?

The average ascent is given by

$$\begin{aligned} \text{avg. ascent} &= \frac{800 - 0}{8 - 11.5} \\ &= -228.57 \text{ ft/hour} \end{aligned}$$

Note that the climber is climbing down, hence the negative rate of ascent.

8. (3 points each) A study is published by a group of researchers at a prominent university that gives a person's expected annual salary after 10 years of work as a function, f , of the total amount of money that that person spent on college tuition. (The group counts loans, scholarships, family contributions, etc., as tuition that a person pays.) The tuition and salary are both measured in thousands of dollars.

(a) What does the statement $f(5) = 20$ mean in practical terms?

The statement $f(5) = 20$ means that the expected annual salary after 10 years of work for a person who spent 5 thousand dollars on college tuition is 20 thousand dollars.

(b) What does $f^{-1}(50) = 20$ mean in practical terms?

The expression $f^{-1}(50) = 20$ tells us that if one wants to have an annual salary of 50 thousand dollars after 10 years of work, then one should spend 20 thousand dollars on college tuition.

(c) What do high-priced private schools hope is true about the sign of f' ? Explain.

A high-priced private school would hope that the more one spends on college tuition, the higher the expected annual salary. Otherwise students would not be willing to pay more money to attend their school. So they would hope the function is an increasing function, i.e., that the derivative of f is positive.

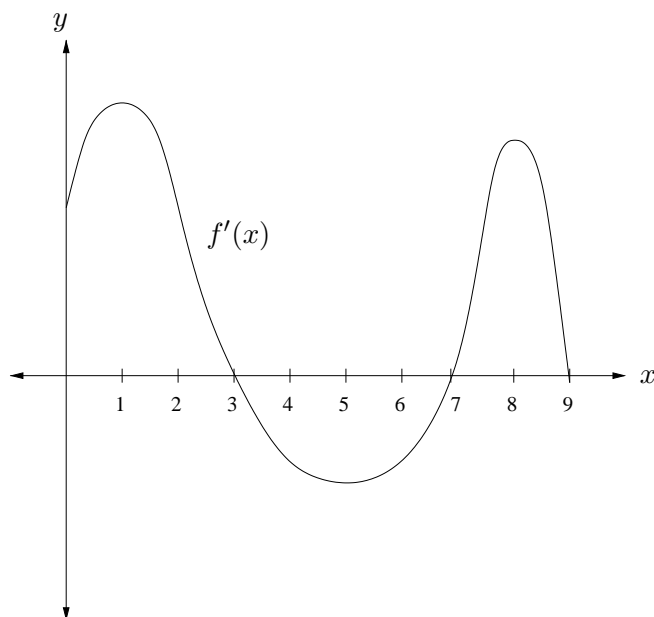
(d) What does the statement $f'(35) = 3$ mean in practical terms?

The expression $f'(35) = 3$ tells us that if a person spends 35 thousand on college, then the expected annual salary after 10 years of work would be approximately 3 thousand dollars more for an additional 1 thousand dollars spent on college tuition.

(e) Suppose you are trying to pick a college and your only concern is your expected salary after 10 years of work. If one of the schools you are considering will cost you 80,000 in tuition and $f'(80) = -0.5$, should you choose a more expensive, less expensive, or that particular school? Justify your answer.

The derivative being negative tells you that the function f is decreasing when one spends \$80 thousand for college tuition. Therefore, you would be wise to attend a school that would cost you less than \$80,000.

9. (10 points) The graph of $f'(x)$ (i.e., the *derivative* of f) is given below. Use the graph to answer the following questions:



(a) For which intervals is f increasing?

f is increasing when its derivative is positive, so for $0 < x < 3$ and $7 < x < 9$.

(b) For which intervals is f'' negative?

f'' is negative when f' is decreasing, so for $1 < x < 5$ and $8 < x < 9$.

(c) For which value(s) of x (if any) does f have a local maximum?

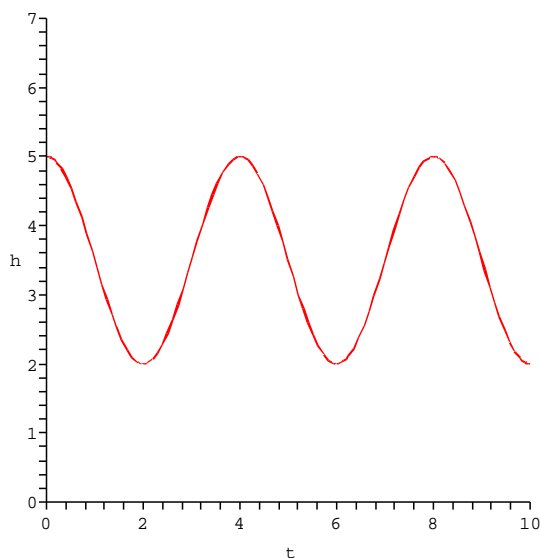
[Note: This was excluded from grading.] f has a local maximum at a value a when f' is positive for $x < a$ and negative for $x > a$. So f has a local maximum when $x = 3$.

(d) For which value(s) of x (if any) does f switch from concave up to concave down?

f will switch from concave up to concave down when the second derivative switches from being positive to being negative, i.e., when the derivative switches from increasing to decreasing, so at $x = 1$ and $x = 8$.

10. (4 points each) At a nearby elementary school the seats of the swing set sit 2 feet off the ground when at rest. While observing a child swing, you note that the seat reaches a maximum height of 5 feet from the ground when the child swings without the aid of pushing from an adult. It takes the child 4 seconds to travel between successive maximum heights. (One is achieved while swinging forward, one while swinging backwards.)

(a) Sketch a graph of the seat's height above the ground (in feet) as a function of time (in seconds) on the axes provided below. Assume that at $t = 0$ the child is at her maximum height, and that she reaches the same maximum height each swing through. Be sure to label the axes carefully! .



(b) Write a trigonometric equation describing the height of the seat as the child swings back and forth.

The highest point will be $h = 5$ and the lowest point will be $h = 2$, so the midline is given by $h = 3.5$. The amplitude is $A = \frac{5-2}{2} = \frac{3}{2}$. The period is 4, and so $B = \frac{\pi}{2}$. Therefore the equation is given as

$$h(t) = \frac{3}{2} \cos\left(\frac{\pi}{2}t\right) + 3.5.$$

(c) At which time(s) during the first 4 seconds of motion is the height of the seat changing most rapidly?

The times when the height of the seat is changing most rapidly occur when the function is the steepest, i.e., when the absolute value of the derivative is at its largest. This occurs at $t = 1$ and $t = 3$ seconds during the first cycle.