MATH 115 — SECOND MIDTERM EXAM

March 30, 2004

NAME: ____

INSTRUCTOR: ____________________ SECTION NO: ________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	14	
2	6	
3	8	
4	8	
5	20	
6	12	
7	10	
8	10	
9	12	
TOTAL	100	

1. (2 points each) Circle "True" if the statement is *always* true. Otherwise, circle "False." No explanation is necessary.

(a) Let f be a continuous function on the interval $[1, 10]$ and differentiable on $(1, 10)$. Suppose that $f(5) = 3$ and $f(2) = 1$. Then there is a point c in the interval $(2, 5)$ so that $f'(c) = \frac{2}{3}$.

True   False

(b) If $g(x) = \frac{1}{f(x)}$, then $g'(x) = -\frac{1}{[f'(x)]^2}$.

True   False

(c) If a is a local maximum for the function f on the interval $[2, 50]$, then $f'(a) = 0$.

True   False

(d) If $g(x) = f^{-1}(x)$, then $g'(x) = (-1)f^{-2}(x)$.

True   False

(e) The 100^{th} derivative of $f(x) = x^5 + e^{2x}$ at $x = 0$ is 2^{100}.

True   False

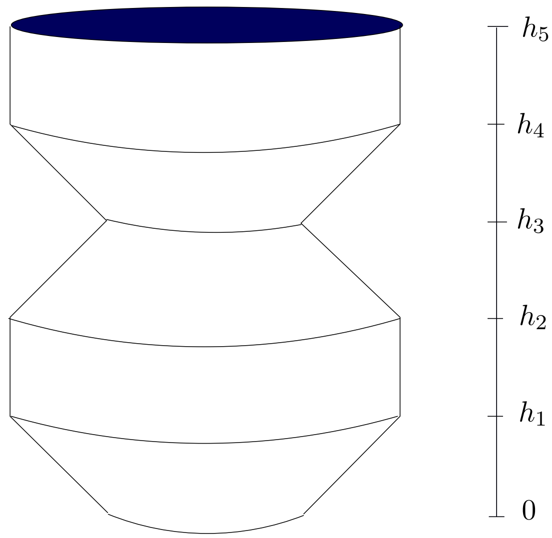
(f) If $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$, then $f'(x) = (x-1)+(x-2)+(x-3)+(x-4)+(x-5)+(x-6)$.

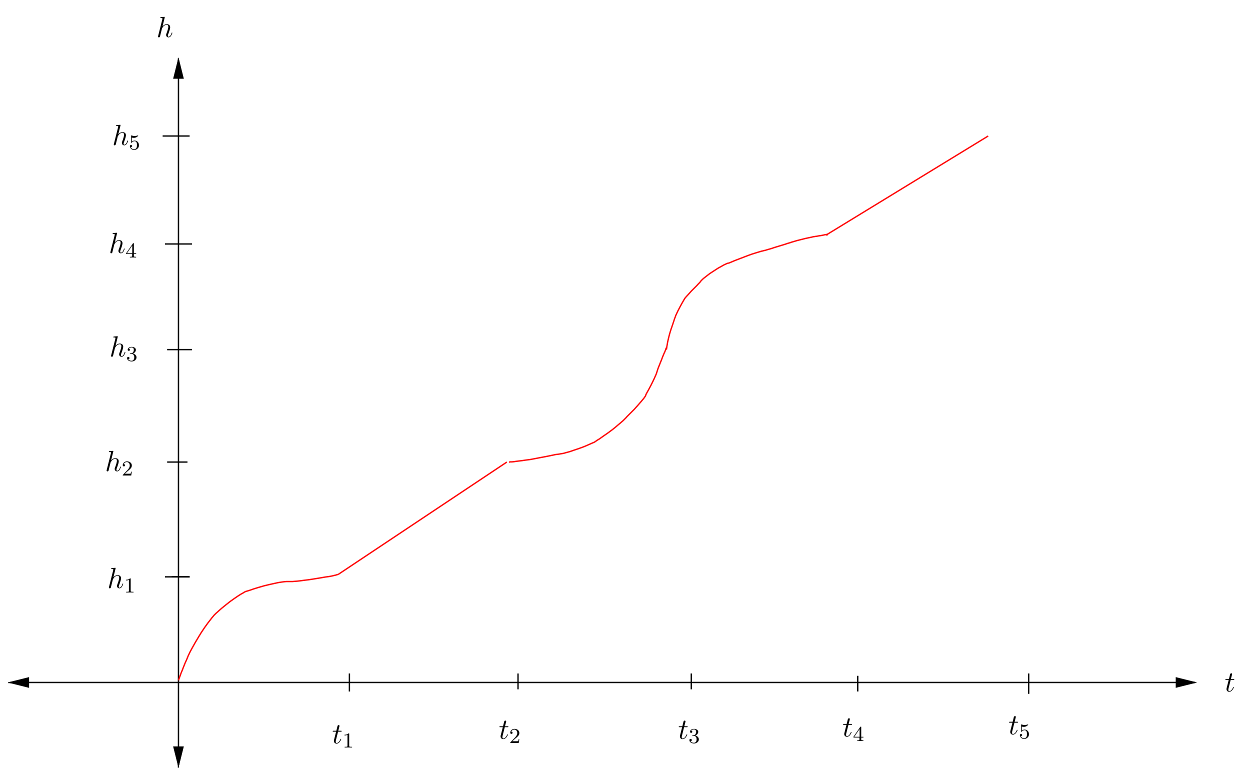
True   False

(g) If f is continuous on $[a, b]$, then f has a global maximum and a global minimum on that interval.

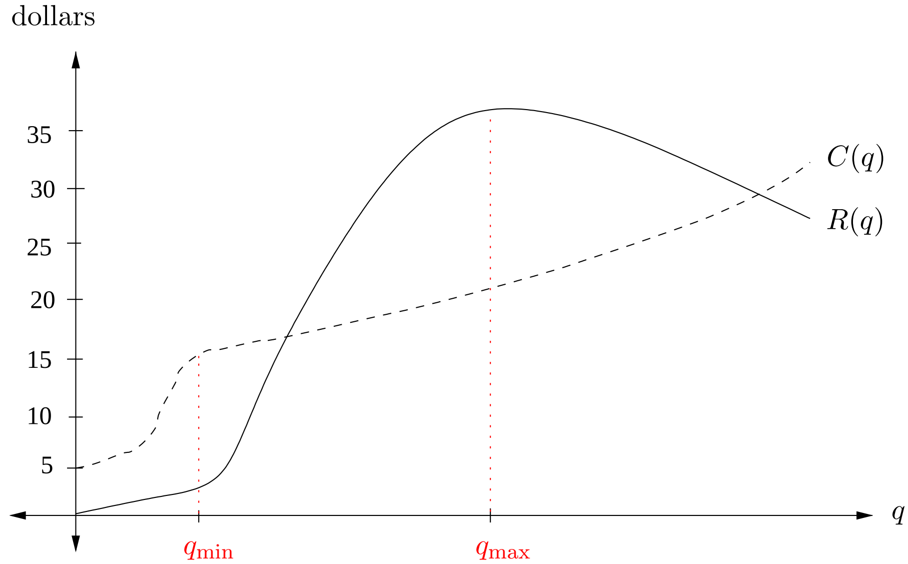
True   False

2. (6 points) Your local coffee shop is running a promotion. They will fill up any shape container you bring to them for 50 cents. You bring in the coffee mug pictured below. If the coffee flows into the container at a constant rate, sketch a graph of the depth of coffee against time. Clearly indicate on the graph the points that correspond to when the coffee reaches heights h_1 through h_5 (indicated to the right of the coffee cup). [Note: On this problem, it's the *shape* of the graph that is important.]





3. (8 points) While home for summer break you find yourself unable to find a steady summer job. You decide to open up a lemonade stand for a day and try to use what you've learned in calculus to your benefit. Below are the graphs of your cost, C, and revenue, R (both measured in dollars), as functions of the number of liters of lemonade you sell, q.



(a) What is the fixed cost of running your lemonade stand?

The fixed costs of running the lemonade stand can be read as the value $C(0)$, which is the cost when selling no lemonade. This is easily seen to be \$5.

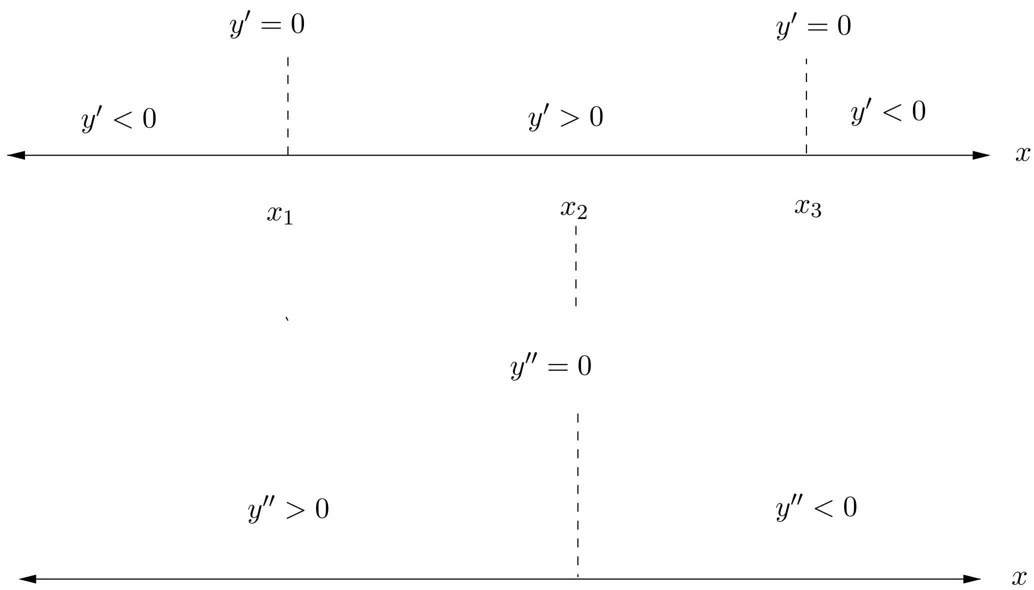
(b) Indicate the point on the q−axis above that maximizes your profit. Label that point q_{max}. Explain how you arrived at your choice of q_{max}.

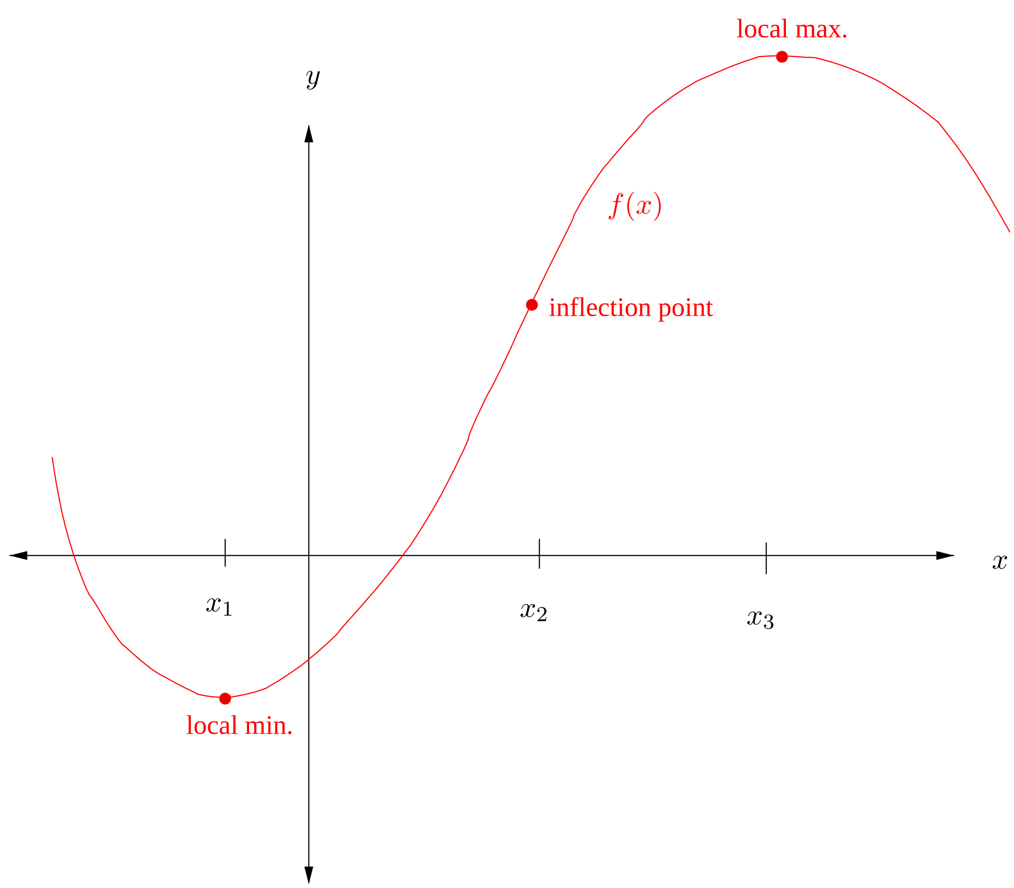
In order to find the point that maximizes profit, one can either look for the q−value that gives the largest value of $R(q) - C(q)$ or one can look for the q−value for which $R'(q) = C'(q)$ and $R(q) > C(q)$.

(c) You decide to run your lemonade stand for another day and put your economically challenged cousin in charge of it for you. Unfortunately, he gets confused and sells the amount of lemonade that will maximize your *losses*. Assuming the graph above is valid for the second day as well, indicate on the q−axis the amount of lemonade your cousin sold. Label this point as q_{min}. Explain how you arrived at your choice of q_{min}.

In this case, one just reverses the roles of C and R in part (b). So one looks to find q for which $C(q) - R(q)$ is largest or the q−value where $R'(q) = C'(q)$ but $C(q) > R(q)$.

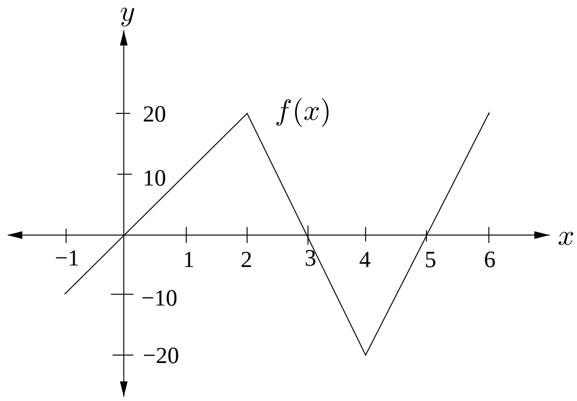
4. (8 points) Sketch a possible graph of $y = f(x)$ using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. Assume the function is defined and continuous for all x. Clearly label all local extrema and inflection points.





Note: The graph may be shifted along the $y-$axis. The shape of the graph must be the same.

5. (20 points) A graph of $y = f(x)$ and a table of values for $g(x)$ and $g'(x)$ are given below. Use them to solve (a)-(d).



x	$g(x)$	$g'(x)$
0	10	-3
1	-2	4
2	5	20

(a) If $h(x) = 2f(x) + x^5$, find $h'(5)$.

$h'(x) = 2f'(x) + 5x^4$. So evaluating this at 5 and using that $f'(5) = 20$ we have $h'(5) = 40 + 5^5 = 3165$.

(b) If $p(x) = 6f(x)(g(x) + 2)$, then find $p'(1)$.

$p'(x) = 6f'(x)(g(x)+2)+6f(x)g'(x)$ using the product and chain rules. Thus $p'(1) = 6f'(1)(g(1)+2) + 6f(1)g'(1) = 6 \cdot 10 \cdot (-2 + 2) + 6 \cdot 10 \cdot 4 = 240$.

(c) If $r(x) = g(f(x) - 9)$, find $r'(1)$.

$r'(x) = g'(f(x) - 9)f'(x)$ using the chain rule. Thus $r'(1) = g'(f(1) - 9)f'(1) = g'(1)f'(1) = 4 \cdot 10 = 40$.

(d) If $j(x) = g(f(3x)) + \cos(\frac{\pi}{2}x)$, then find $j'(1)$.

$j'(x) = g'(f(3x))f'(3x)3 - \sin(\frac{\pi}{2}x)\frac{\pi}{2}$ by repeated applications of the chain rule. Thus $j'(1) = g'(f(3))f'(3)3 - \frac{\pi}{2}\sin(\frac{\pi}{2}) = -3 \cdot (-20) \cdot 3 - \frac{\pi}{2} = 180 - \frac{\pi}{2}$.

6. (12 points) The electric field (in Newtons/Coulomb) outside of a charged sphere of charge q (in Coulombs) is given by the formula

$$E(r) = \frac{kq}{r^2}$$

where k is a positive constant and r is the distance measured in meters from the center of the sphere to the point from which one is measuring.

(a) Find a formula for the local linearization of $E(r)$ near $r = 2$ meters. [Your answer will contain k and q.]

Denote the local linearization of $E(r)$ by $L(r)$. Then note that $E'(r) = -\frac{2kq}{r^3}$. Thus we have that $E'(2) = -\frac{kq}{4}$ and $E(2) = \frac{kq}{4}$, which is all the information we need to determine the local linearization. So $L(r) = \frac{kq}{4} - \frac{kq}{4}(r-2)$.

(b) Use your result from part (a) to approximate $E(2.1)$. [Again, your answer will contain k and q.]

The local linearization provides an approximation to the function near the point 2 in this case. Therefore we have $E(2.1) \approx L(2.1) = \frac{kq}{4} - \frac{kq}{4}(2.1-2) = \frac{9kq}{40}$.

(c) Assuming $q > 0$, do you expect your estimate in part (b) to be an over- or underestimate of the actual value of $E(2.1)$? Use calculus to justify your answer. Explain.

We use the concavity of the function $E(r)$ to answer this question. Note that $E''(2) = \frac{6kq}{2^4} > 0$ since k, q are both positive constants. Thus the function is concave up at the point 2. Looking at a graph of a concave up function it is easy to see that the tangent line is an underapproximation in this case. Therefore our estimate should be an underapproximation to the actual value $E(2.1)$.

7. (10 points) While exploring an exotic spring break location, you discover a colony of geese who lay golden eggs. You bring 20 geese back with you. Suppose each goose can lay 294 golden eggs per year. You decide maybe 20 geese isn't enough, so you consider getting some more of these magical creatures. However, for each extra goose you bring home there are less resources for all the geese. Therefore, for each new goose the amount of eggs produced will decrease by 7 eggs per goose per year. How many more geese should you bring back if you want to maximize the number of golden eggs per year laid? Show your work.

Let x be how many more geese you bring back. So the total number of geese you have is $20 + x$. The number of eggs a single goose can lay in a year is given by $294 - 7x$. Therefore, the total number of eggs produced is

$$A(x) = (20 + x)(294 - 7x).$$

This is the function we would like to maximize. So we take its derivative:

$$\begin{aligned} A'(x) &= 1(294 - 7x) + (20 + x)(-7) \\ &= 154 - 14x \end{aligned}$$

Setting this equal to 0 to find the critical points we obtain $0 = 154 - 14x$. Thus the only critical point is at $x = 11$. To check that this is actually a maximum, we take the second derivative of $A(x)$ which is $A''(x) = -14$, so this is a maximum. Therefore we should bring back 11 more geese to maximize the number of golden eggs produced each year.

8. (10 points) The ideal gas law relates the volume and pressure of a gas to the temperature of the gas. The formula can be given as

$$PV = cT$$

where P is the pressure of the gas measured in *atmospheres*, V is the volume of the gas measured in *liters*, c is a positive constant, and T is the temperature of the gas measured in *kelvins*. (Remember, a temperature measured in kelvins is always positive!)

(a) If T is held constant, find $\frac{dV}{dP}$.

Assuming $P \neq 0$ we can rewrite the equation as $V = \frac{cT}{P}$. Now we just use the power rule to obtain $\frac{dV}{dP} = -\frac{cT}{P^2}$.

(b) What is the meaning of the sign of your answer to part (a)? Explain this in everyday terms.

Noting that c, T, and P are all positive quantities, part (a) tells us that $\frac{dV}{dP}$ is negative. This says that V is a decreasing function of P. What this means in everyday terms is that as one increases the pressure on a gas, the volume of the gas goes down.

(c) Suppose V, P, and T are all functions of the time t. Find $\frac{dT}{dt}$.

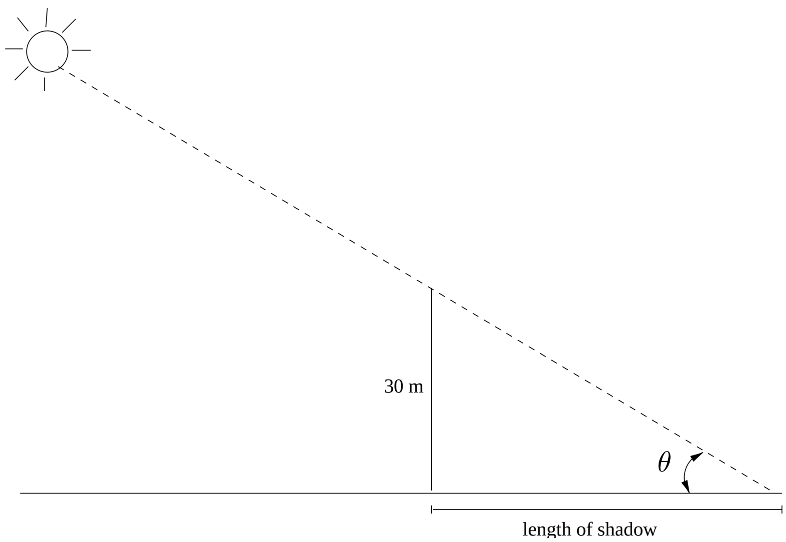
Taking the derivative with respect to t on each side of the equation we have:

$$V\frac{dP}{dt} + P\frac{dV}{dt} = c\frac{dT}{dt}.$$

Thus, we have

$$\frac{dT}{dt} = \frac{1}{c}\left(V\frac{dP}{dt} + P\frac{dV}{dt}\right).$$

9. (12 points) On a spring day the morning sun is rising at the rate of $\frac{12\pi}{180}$ radians per hour. How fast is the shadow cast by a building that is 30 meters high changing when the sun is $\frac{\pi}{4}$ radians above the horizon in the morning? The following picture may be helpful.



Let x denote the length of the shadow. Then we have the equation $\tan(\theta) = \frac{30}{x}$ relating θ to x. Since we are interested in $\frac{dx}{dt}$, we differentiate this formula with respect to t and obtain:

$$\frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = -\frac{30}{x^2}\frac{dx}{dt}.$$

Now we are interested in finding $\frac{dx}{dt}$ when $\theta = \frac{\pi}{4}$, so we need to find x, $\frac{1}{\cos^2(\frac{\pi}{4})}$, and $\frac{d\theta}{dt}$ in this case. x can be found from our original equation to be $x = \frac{30}{\tan(\frac{\pi}{4})} = 30$ meters. $\frac{1}{\cos(\frac{\pi}{4})} = \frac{2}{\sqrt{2}}$. So $\frac{1}{\cos^2(\frac{\pi}{4})} = 2$. We are given that $\frac{d\theta}{dt} = \frac{12\pi}{180}$ regardless of the value of θ. So now we just solve for $\frac{dx}{dt}$ and plug in the values we have just found to conclude that

$$\begin{aligned}\frac{dx}{dt} &= -\frac{30^2}{30}\frac{1}{\cos^2(\frac{\pi}{4})}\frac{d\theta}{dt} \\ &= -30 \cdot 2 \cdot \frac{12\pi}{180} \\ &= -12.57\,\mathrm{m/hr}.\end{aligned}$$

Thus the length of the shadow is decreasing at 12.57 meters per hour when the sun is $\frac{\pi}{4}$ radians above the horizon in the morning.