MATH 115 — FINAL EXAM

April 26, 2004

NAME: SOLUTION KEY

INSTRUCTOR: ____________________ SECTION NO: ____________________

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	14	
2	4	
3	8	
4	10	
5	12	
6	10	
7	13	
8	10	
9	8	
10	11	
TOTAL	100	

1. (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only is the statement is *always* true. No explanation is necessary.

(a) Suppose f is a continuous function such that $f(1) = 5$ and $f'(x) < 0$ for $x \geq 5$. Then there is an $x > 5$ so that $f(x) = 0$.

True False

(b) $\int_0^{10} f(x)dx$ is a function of x.

True False

(c) Let

$$f(x) = \begin{cases} 5 & 0 \leq x < 2 \\ 0 & 2 \leq x < 8 \\ 10 & 8 \leq x \leq 10. \end{cases}$$

Then the average value of $f(x)$ on $[0, 10]$ is 3.

True False

(d) If f' is continuous and has a local maximum at a, then f has an inflection point at a.

True False

(e) $\int x \ln(x)dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$

True False

(f) A function can have more than one antiderivative.

True False

(g) For a continuous function f, either the left-hand sum or the right-hand sum is an overestimate of the definite integral of f on an interval $[a, b]$.

True False

2. (4 points) As an avid online music trader, your rate of transfer of mp3's is given by $m(t)$ measured in songs/hour where $t = 0$ corresponds to 5 pm. Explain the meaning of the quantity

$$\int_0^5 m(t)\,dt.$$

$\int_0^5 m(t)\,dt$ represents the number of songs transferred between 5 pm and 11 pm.

3. (8 points) Suppose $\int_{-3}^{4} f(x)dx = 10$, $\int_0^4 f(x)dx = 2$, and that f is an ***odd*** function. For each of the following integrals fill in the answer in the space provided.

(a) $\int_{-3}^{4} 6f(x)\,dx = 6\int_{-3}^{4} f(x)\,dx = 60$

(b) $\int_{-3}^{0} f(x)\,dx = \int_{-3}^{4} f(x)\,dx - \int_0^4 f(x)\,dx = 8.$

(c) $\int_{-4}^{0} f(x)\,dx = -\int_0^4 f(x)\,dx = -2$ where we use that $f(x)$ is an odd function.

(d) $\int_{-4}^{-3} f(x)\,dx = \int_{-4}^{0} f(x)\,dx - \int_{-3}^{0} f(x)\,dx = -10$

4. (10 points) Let f be a continuous differentiable function of x. Suppose f is always increasing. The following is a table of values of $f(x)$.

x	.8	.9	1	1.1	1.2	1.3	1.4	1.5
$f(x)$	3	25	26	27	49	52	62	63

(a) Using the table above, give an approximation of $f'(1)$.

One can use several different points to get an approximation of $f'(1)$. For example,

$$f'(1) \approx \frac{f(1.1) - f(1)}{1.1 - 1} = \frac{27 - 26}{1.1 - 1} = 10.$$

(b) Would a left-hand or a right-hand sum give a lower estimate of $\int_1^{1.5} f(x)\,dx$? Why?

Since $f(x)$ is an increasing function, the left-hand sum will give a lower estimate of $\int_1^{1.5} f(x)\,dx$.

(c) Using the table above, give upper and lower estimates of $\int_1^{1.5} f(x)\,dx$.

As we determined in part (b), the left-hand sum gives a lower estimate of $\int_1^{1.5} f(x)\,dx$ and similarly the right-hand sum gives an over estimate.

$$\begin{aligned} LHS &= 0.1(26 + 27 + 49 + 52 + 62) \\ &= 21.6 \end{aligned}$$

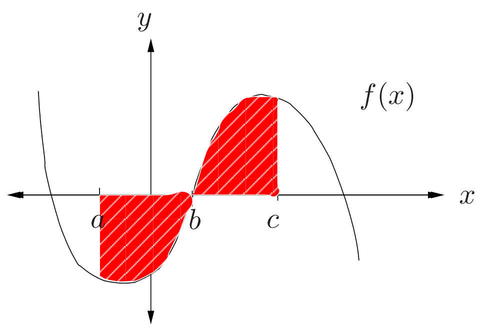
$$\begin{aligned} RHS &= 0.1(27 + 49 + 52 + 62 + 63) \\ &= 25.3 \end{aligned}$$

So we have

$$21.6 \leq \int_1^{1.5} f(x)\,dx \leq 25.3.$$

5. (12 points) For parts (a) - (c), on the graphs below, show a graphical interpretation for each of the given expressions, and then explain how the quantities given by the expression relate to your drawings on the graphs.

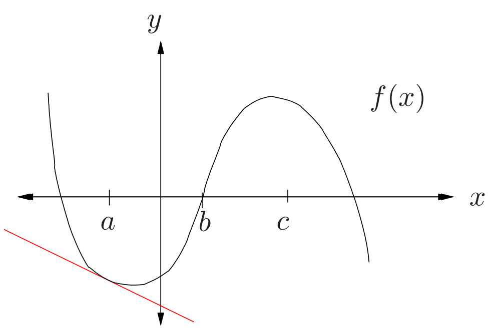
(a) $\int_a^c f(x)\,dx$



$\int_a^c f(x)\,dx$ is the area between $f(x)$ and the $x-$axis from $x = a$ to $x = c$.

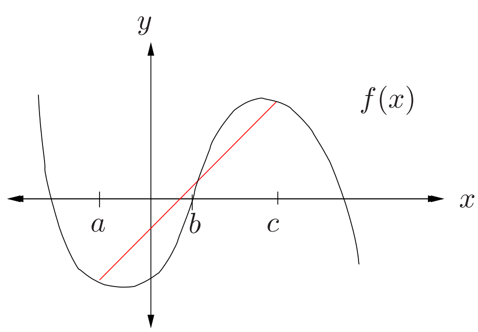
h

(b) $f'(a)$



$f'(a)$ is the slope of the tangent line to $f(x)$ at $x = a$.

(c) $\dfrac{f(c) - f(a)}{c - a}$



This is the slope of the line connecting the points $(a, f(a))$ and $(c, f(c))$.

6. (10 points) It is estimated that the rate people will visit a new theme park is given as

$$r(t) = \frac{A}{1 + Be^{-0.5t}}$$

where A and B are both constants and $r(t)$ is measured in people/day, and $t = 0$ corresponds to opening day.

(a) Write an integral that gives the total number of people visiting the park in the first year it is open. Do not try to evaluate the integral!

The total number of people visiting the park in the first year is $\int_0^{365} r(t)\,dt$.

(b) Suppose that $A = 100$ and $B = 5$. Given that

$$\frac{d}{dt}\left(2A\ln(1 + Be^{-0.5t}) - 2A\ln(Be^{-0.5t})\right) = \frac{A}{1 + Be^{-0.5t}},$$

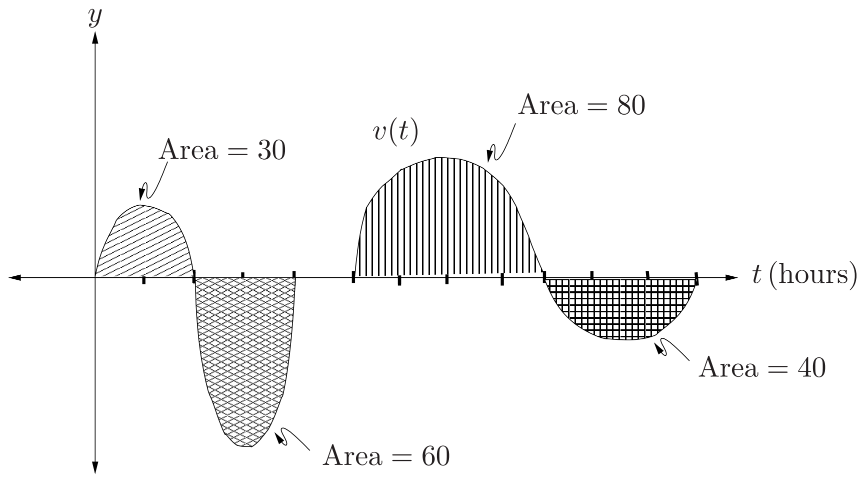
use the First Fundamental Theorem of Calculus to evaluate how many people visit the park during the first year it is open. Make sure you clearly indicate your use of the theorem.

Let $R(t) = 200\ln(1 + 5e^{-0.5t}) - 200\ln(5e^{-0.5t})$. Then $R'(t) = r(t)$, so the First Fundamental Theorem of Calculus says that

$$\begin{aligned}\int_0^{365} r(t)\,dt &= R(365) - R(0)\\ &= 36,141.\end{aligned}$$

So $36,141$ people visit the theme park during its first year.

7. (13 points) The following graph gives a taxi driver's velocity (in miles per hour) as a function of time. Assume the driver only travels on a straight road east and west. Positive velocity indicates travel to the east, negative velocity indicates travel to the west. Assume the driver starts his day at the airport at 6 am when $t = 0$, and that the intervals between each tick mark on the horizontal axis correspond to one hour. The area of each shaded region is indicated on the graph.



(a) At approximately what time(s) is the driver's acceleration 0?

The driver's acceleration is 0 when $v'(t) = 0$. This occurs approximately at the times: 7 am, 9 am, 10-11 am, 1 pm, and 4:30 pm

(b) If the taxi driver takes a break at 10 am, how far is he from the airport? Be sure to note whether he is east or west of the airport. Justify your answer appropriately.

The driver will be 30 miles west of the airport. The distance he has travelled is $\int_0^4 v(t)\,dt$, which is $30 - 60 = -30$. The negative indicates he is west of the airport.

(c) At what time is the driver the furthest from the airport? How far away is he at this time?

One can tell the driver's distance from the airport at time T by evaluating the integral $\int_0^T v(t)\,dt$. Doing this ones see that the driver is furthest from the airport at 3 pm and he is 50 miles east of the airport.

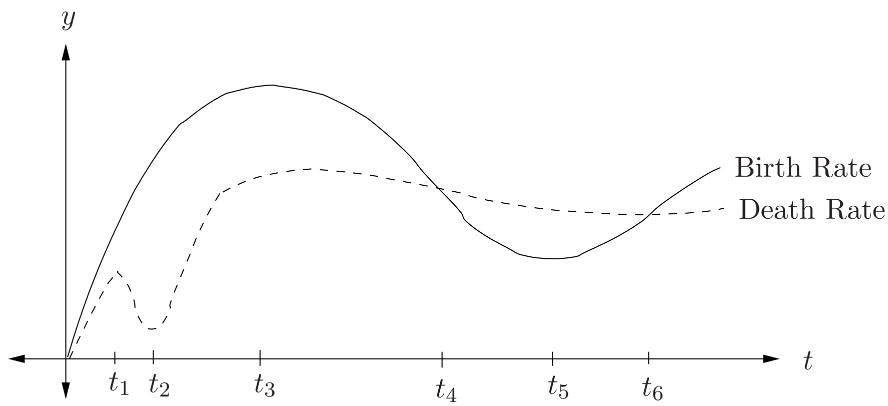
(d) How many times after 6 am during the day does the driver pass the airport?

Again, looking at the distance the driver is from the airport we look for times when the distance is zero. This will occur sometime between 8 am and 10 am and then again between 11 am and 3 pm. After that his distance from the airport will always be positive to the east. Therefore the driver passes the airport 2 times during the day.

8. (10 points) You have given up on your lemonade stand after your cousin ran it into the ground. However, you still need to make some money over the summer so you decide to tutor local high school students in mathematics. You start off charging \$45 per hour. Only 2 students are willing to pay this rate for your expert knowledge. However, you find that for each \$3 less per hour that you charge, 1 more student is willing to sign up for tutoring. You decide you can tutor for a maximum of 15 hours per week, that you will meet with each student one hour per week, and that you will only tutor one student at a time. What should you charge if making the most money per week is your only goal? In order to get full credit, you must use the techniques of calculus to solve this problem and show all of your work!

Let x be the number of additional students to the 2 you would have at \$45 per hour. So the total number of students you have is $2 + x$ and the amount you charge each student is $\$45 - 3x$. The total amount of money you will make each week is $A(x) = (45-3x)(2+x)$. To find out how much you should charge, you need to maximize $A(x)$. $A'(x) = 39 - 6x$, so setting this equal to zero we see we have a critical point at $x = 6.5$. $A''(6.5) = -6 < 0$ so this is a local maximum. However, we can't tutor half of a student, so we can either tutor 8 students or 9 students corresponding to $x = 6$ or $x = 7$. $A(6) = \$216$ and $A(7) = \$216$, so the best option would be to choose to $x = 6$ to make the same amount of money with less work. Now we need to check the endpoints of our interval which occur when you have 0 students or 15 students. If you have 0 students you clearly will make no money, so this can't be the maximum of $A(x)$ since our local maximum gives \$216. The other endpoint gives $A(13) = \$90$. So our maximum does occur when $x = 6$. Therefore, we should charge $\$45 - 3(6) = \$27$ per hour.

9. (8 points) Last year a local entomologist studied the birth and death rate of mosquitos in the Ann Arbor area during the month of May. His research yielded the following graph.



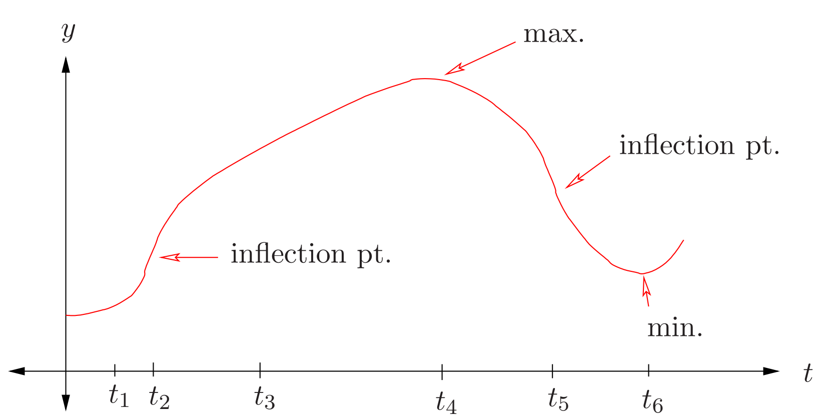
(a) Which of the labelled times t_1 through t_6 is the time when there were the largest number of mosquitos in Ann Arbor during May?

The largest number of mosquitos in Ann Arbor during May occurred at $t = t_4$. Up to this point more mosquitos are being born then die off, so our number of mosquitos is increasing. Between t_4 and t_6 more are dying then being born, so we are losing mosquitos. After t_6 we are gaining mosquitos again, but since the area between the Death Rate and Birth Rate is greater from t_4 to t_6 then after t_6, we still have less mosquitos then we did at $t = t_4$.

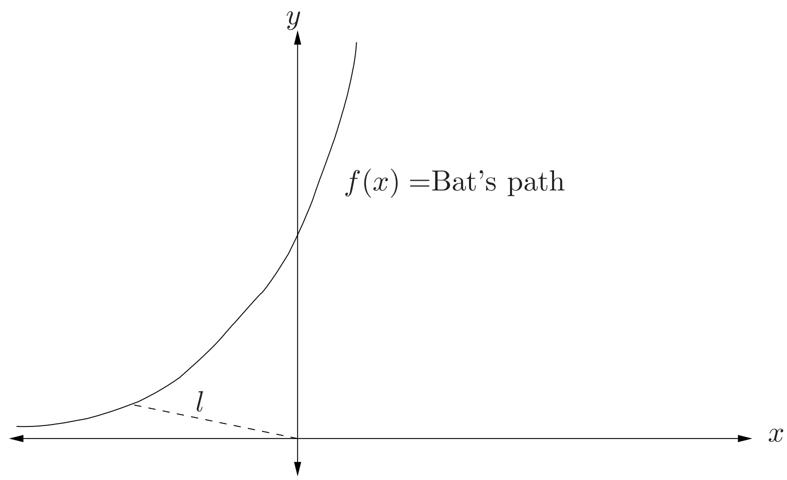
(b) Which of the labelled times t_1 through t_6 is the time when the quantity of mosquitos in Ann Arbor was increasing most rapidly during May?

The quantity of mosquitos is increasing most rapidly when there is the greatest difference between the Birth Rate and the Death Rate. This occurs when $t = t_2$.

(c) Sketch a possible graph of the number of mosquitos alive during the month of May on the axes below. Make sure to clearly indicate any maxima, minima, or inflection points.



10. (11 points) Hiking through the forest you come upon a cave. As you stand outside the cave and peer in, a bat flies out towards you before veering off into the forest. The bat's path is given in the figure below where the origin represents where you are standing. The distance l represents the distance between you and the bat. Everything is measured in feet.



(a) Find a formula for l^2 in terms of x and $f(x)$.

Using the Pythagorean Theorem one gets

$$l^2 = x^2 + f(x)^2.$$

(b) Let $D = l^2$ and find $\frac{dD}{dx}$.

$$\frac{dD}{dx} = 2x + 2f(x)f'(x)$$

(c) The minimum distance between you and the bat occurs when D is minimized. Find the value of x at this point in terms of $f(x)$ and $f'(x)$.

Setting our answer in part (b) equal to 0, we get that

$$x = -f(x)f'(x)$$

is a critical point. To show this is a minimum, we take the second derivative

$$\frac{d^2D}{dx^2} = 2 + 2\left[f'(x)\right]^2 + 2f(x)f''(x).$$

We see from the graph that everything here is positive, so the second derivative of $D(x)$ is positive, so we have indeed found a minimum.

(d) Suppose $f(x) = e^{x+3}$. If a bat comes with 5 feet of you, a panic attack will occur. (Remember that the distance between you and the bat is l, not D!) Did the bat induce a panic attack? [Hint: You are ***encouraged*** to use your calculator here!]

Using what we found in part (c) we have

$$\begin{aligned} x &= -e^{x+3}e^{x+3} \\ &= -e^{2x+6}. \end{aligned}$$

So we need to solve the equation $x + e^{2x+6} = 0$ for x. We use the calculator to do this, getting the answer $x \approx -2.53$. So then $f(-2.53) = e^{-2.53+3} \approx 1.60$. Looking back at our equation for l^2, we have $l^2 \approx 8.96$, so $l \approx 3$ft. Therefore a panic attack does incur!