MATH 115 — FIRST MIDTERM EXAM

February 8, 2005

NAME: SOLUTION KEY

INSTRUCTOR: ______________________ SECTION NO: __________

1. Do not open this exam until you are told to begin.
2. This exam has 8 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and other sound devices, and remove all headphones.

PROBLEM	POINTS	SCORE
1	12	
2	12	
3	16	
4	15	
5	8	
6	5	
7	14	
8	18	
TOTAL	100	

1. (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only if the statement is *always* true. No explanation is necessary.

(a) Every continuous function is differentiable.

True FALSE

(b) If $f'(x) > 0$ for all x in the interval (a, b), then f is increasing on the interval (a, b).

TRUE False

(c) By definition, the instantaneous velocity is equal to a difference quotient.

True FALSE

(d) Every rational function has a vertical asymptote.

True FALSE

(e) If a function is not continuous at a point, then it is not defined at that point.

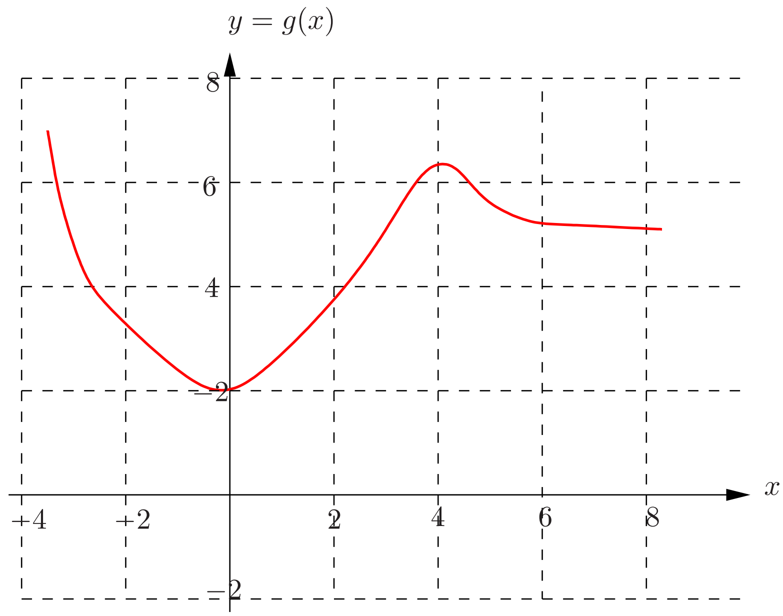
True FALSE

(f) If a function f is decreasing on an interval, then f' is decreasing on that interval.

True FALSE

2. (7+2+3 points) **(a)** On the axes below, sketch a graph of a single, continuous, differentiable **function**, g, with **all** of the following properties.

- $g(0) = 2$
- g' is negative for $x < 0$ and $x > 4$
- g is increasing for $0 < x < 4$
- g'' is positive for $x < 3$
- $g'(4) = 0$
- $g(x) \to 5$ as $x \to \infty$



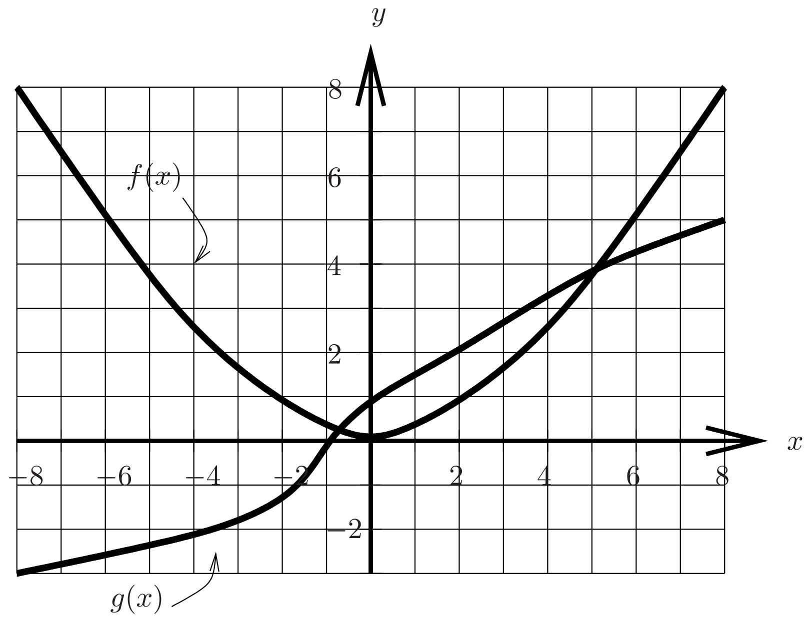
(b) What is $\lim_{x \to -\infty} g(x)$?

$+\infty$

(c) If $g'(1) = 1/2$, is it possible to have $g'(2) = 1/4$? Explain.

No. Since g is increasing for $0 < x < 4$ and concave up for $x < 3$, if $g'(1) = 1/2$ then $g'(2) > 1/2$.

3. (16 points) The graphs of two functions, f and g are given in the figure below.



Use the graphs to estimate or answer each of the following.

(a) $f(g(2))$ 1

(b) $3g(-4)$ -6

(c) $g(f(2))$ 1.5

(d) $g(2) \cdot f(6)$ 10

(e) If $h(x) = f(x-1)$, how is the graph of h related to the graph of f?

The graph of h is the graph of f shifted one unit to the right.

(f) If $j(x) = g(-x)$, how is the graph of j related to the graph of g?

The graph of j is the graph of g reflected over the $y - axis$

(g) The **sign** of $f'(-4)$ is $-$.

(h) The **sign** of $f''(-4)$ is $+$.

4. (6+2+3+4 points) For Valentine's Day last year, Hannah brought her officemates chocolate. As the day progressed, the amount of chocolate remaining in the office decreased. After two hours, there were only 5 pounds of chocolate remaining, and after seven hours, there was only 1 pound left.

(a) Assuming that the amount of chocolate in the office decreased linearly, write an equation for the amount of chocolate, c in pounds, left after t hours.

$c(2) = 5$ and $c(7) = 1$.

$$\text{slope} = \frac{\text{change in chocolate}}{\text{change in time}} = \frac{5-1}{2-7} = \frac{-4}{5}.$$

Thus, we have that $c(t) = \frac{-4}{5}t + b$.

To solve for b, substitute into the equation one of the given points: $\frac{-4}{5}(2) + b = 5$.

This gives $b = \frac{33}{5}$ and $c(t) = \frac{-4}{5}t + \frac{33}{5}$.

(b) How much chocolate did Hannah bring to the office?

The amount of chocolate that Hannah brought to the office is given by $c(0)$. Plugging in zero for t into the equation from (a) gives $\frac{33}{5}$ or 6.6 pounds of chocolate.

(c) What is the practical interpretation of the slope of your linear function in the context of this problem?

Every five hours, Hannah's officemates eat four pounds of chocolate.

(d) Now, assume instead that the amount of chocolate left at time t was represented by the exponential function $C(t) = ab^t$, find a and b, and express your answer as a function. [Do not assume that this function indicates the same beginning amount of chocolate as in part (a). Use the data given in the original statement of the problem to determine a and b.]

$c(2) = 5$ and $c(7) = 1$

$\Rightarrow ab^2 = 5$ and $ab^7 = 1$

$$\Rightarrow \frac{ab^2}{ab^7} = \frac{5}{1}$$

$$\Rightarrow b^{-5} = 5$$

$$\Rightarrow b = \frac{1}{5}^{\frac{1}{5}} \approx .725$$

$$\Rightarrow a(.725)^2 = 5$$

$$\Rightarrow a = 9.5$$

$$\Rightarrow c(t) = 9.5(.725)^t.$$

5. (8 points) After a dementor attack, Harry, the wizard, eats chocolate in order to feel better. When a wizard eats chocolate, the chocolate enters their bloodstream instantaneously and the body metabolizes and eliminates it from the bloodstream at the rate of 20% per hour.

(a) If Harry ate 1/2 pound of chocolate, write a formula for the amount of chocolate, Q (in pounds), remaining in his bloodstream t hours after he ate the chocolate.

Harry ate a 1/2 pound of chocolate, so $Q_0 = 1/2$. We're told that his body metabolizes and eliminates it from the bloodstream at the rate of 20% per hour. Since the function is changing at a constant *percent*, we have an exponential function with the growth factor of $(1 - .20) = .80$ Thus,

$$Q(t) = \frac{1}{2}(0.8)^t.$$

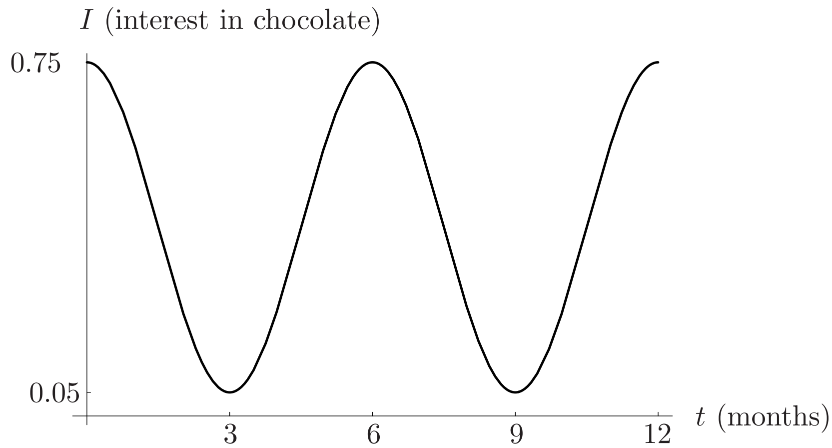
(b) If Harry's chocolate level in his bloodstream becomes lower than 0.2 pounds, he will go into shock. What is the maximum amount of time, t, that he can wait before eating more chocolate? Show your work.

The maximum value of time, t, that Harry can wait is the value of t that satisfies the equation $0.2 = \frac{1}{2}(0.8)^t$.
$\Rightarrow 0.4 = (0.8)^t$
$\Rightarrow ln(0.4) = t * ln(0.8)$
$\Rightarrow t = \frac{ln(0.4)}{ln(0.8)}$
So, $t \approx 4.1$hours.

6. (5 points) Let $f(x) = \sin(3x^2)$. Use the **definition** of the derivative to express $f'(2)$ as a limit. You do **not** need to simplify your expression or try to approximate $f'(2)$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0} \frac{\sin(3(2+h)^2)-\sin(12)}{h}$$

7. (14 points) Kevin's interest in chocolate fluctuates during the year. His girlfriend works at a chocolate factory, and sometimes he gets a chocolate overload. Even at the best of times, he wouldn't gauge his level of "chocolate interest" as more than, say, a 75% interest. Assume that Kevin's interest in chocolate is given by the graph of $I(t)$ shown below, where t is in months and $t = 0$ is January 1st, 2004.



(a) Assuming I is a trigonometric function, find a formula for I in terms of t.

midline: $I = 0.4$
amplitude: $A = 0.35$
period $= 6 \Rightarrow B = \frac{2\pi}{6} = \frac{\pi}{3}$
So, one possible equation for I is

$$I(t) = 0.35\cos(\frac{\pi}{3}t) + 0.4.$$

(b) List all months of 2004 in which Kevin's interest in chocolate was increasing.

Notice that the function is increasing between $t = 3$ and $t = 6$ as well as between $t = 9$ and $t = 12$. We're told in the problem that $t = 0$ corresponds to January 1, 2004. So, the function is increasing during the following months:

April, May, June and October, November, December.

(c) For what value(s) of t during 2004 was Kevin's interest in chocolate increasing the fastest?

$t = 4.5$ and $t = 10.5$

8. (3 points each) Sarah decided to run a marathon. However, she started off way too fast and so her speed decreased throughout the race. Below is a table showing how many miles she had run at time t minutes since the beginning of the race.

time (min)	30	60	90	120	150	180	210	240
distance (miles)	5	9	12.5	15.5	18.5	21	23.25	25.2

Let s be the function such that $s(t)$ is Sarah's distance from the starting line t minutes after the race began.

(a) What is the practical interpretation of $s'(120)$ in the context of this problem?

The expression $s'(120)$ gives Sarah's speed, in miles/min, two hours into the race.

(b) Estimate $s'(120)$.

We must estimate the instantaneous rate of change at $t = 120$. You can estimate this value "from the left", "from the right", or "both the left and the right and take their average". In this case, all will give you the same answer. Here, we estimate from the right:

$$s'(120) \approx \frac{18.5 - 15.5}{150 - 120} = 0.1\text{miles/min.}$$

(c) What is the practical interpretation of $s^{-1}(14)$ in the context of this problem?

The expression $s^{-1}(14)$ gives the length of time that Sarah has been running when she is 14 miles into the race.

(d) Estimate $s^{-1}(14)$.

Notice that at $t = 90$, Sarah is at the 12.5 mile mark and at $t = 120$, Sarah is at the 15.5 mile mark. So, sometime between $t = 90$ and $t = 120$ Sarah passes the 14 mile mark. Since the average speed between mile 12.5 and 15.5 is approximately 1/10 miles/min, to go the 1.5 miles would take approximately 15 minutes. Thus, $s^{-1}(14)$ is approximately equal to $90 + 15 = 105$ minutes.

(e) What does the derivative of $s^{-1}(P)$ at $P = 14$ represent in the context of this problem?

Once Sarah has run 14 miles, $(s^{-1})'(14)$ gives the approximate amount of time it will take her to run the next mile (assuming that her pace stays the same for the next mile). The units of $(s^{-1})'(14)$ are in minutes/mile.

(f) Estimate the derivative of $s^{-1}(P)$ at $P = 14$.

$$(s^{-1})'(14) \approx \frac{120-105}{15.5-14} = 10 \text{ minutes/mile}$$