MATH 115 — Second MIDTERM EXAM

March 29, 2005

NAME: SOLUTION KEY

INSTRUCTOR: ______________________ SECTION NO: __________

1. **Do not open this exam until you are told to begin.**
2. This exam has 8 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of one 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and other sound devices, and remove all headphones.

PROBLEM	POINTS	SCORE
1	8	
2	10	
3	16	
4	12	
5	14	
6	10	
7	10	
8	20	
TOTAL	100	

1 . (8 points) The following table gives values of a continuous, differentiable function f' (i.e., the derivative of f). The statements below the table concern f. For each answer, give the smallest interval that is indicated by the table.

x	-4	-3	-2	-1	0	1	2	3	4
$f'(x)$	3	4	3	2	-1	-7	-2	4	6

(a) The function f has a local minimum between $x =$ __2__ and $x =$ __3__.

(b) The function f has a local maximum between $x =$ __-1__ and $x =$ __0__.

(c) The function f has an inflection point between $x =$ __-4__ and $x =$ __-2__. (There is more than one possible answer here.) (There is another between x=0 and x=2.)

2. (10 points) Let g be a function such that $g(2) = 4$ and whose derivative is known to be $g'(x) = \sqrt{x^2 + 2}$.

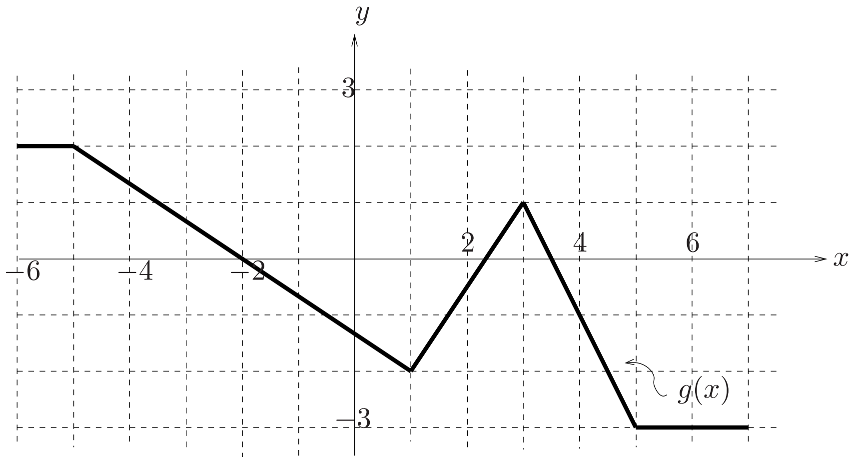
(a) Use a linear approximation to estimate the value of $g(1.95)$. Show your work.

We know that $g(2 + \Delta x) \approx g'(2)(\Delta x) + g(2)$. Since we're looking for $g(1.95)$, we set $\Delta x = -0.05$. Also, $g'(2) = \sqrt{2^2 + 2} = \sqrt{6}$. Plugging these values into the above formula gives $g(1.95) \approx \sqrt{6}(-0.05)+4 \approx 3.878$.

(b) Do you think your estimate in part (a) is an overestimate or an underestimate? Explain.

We first calculate $g''(x) = \frac{1}{2}(2x)(x^2+2)^{\frac{-1}{2}} = \frac{x}{\sqrt{x^2+2}}$. So, g'' is positive for all $x > 0$. Thus, the estimate in part (a) is an underestimate since the tangent line to the graph of g at $x = 2$ lies below the actual graph of g.

3. (16 points) Some values for a differentiable function f are given in the table below, and the graph of $y = g(x)$ on the interval [-6,7] is given in the figure below. Do not assume any information about f or g other than what is given.



x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	0.5	4	7.5	10	9	5	0	3	9
$f'(x)$	3	4	3	2	-1	-7	-2	4	6

Use the table and the graph to find the following, if possible. If any information is missing, explain *clearly* what is missing. Show your work.

(a) Find $h'(4)$ if $h(x) = g(x)f(x)$.

We know $h'(x) = g'(x)f(x) + g(x)f'(x)$. So, $h'(4) = g'(4)f(4) + g(4)f'(4)$. From the table, we can see that $f(4) = 9$ and $f'(4) = 6$. From the graph of g, we can see that $g(4) = -1$. Computing the slope of the line at $g = 4$ gives $g'(4) = -2$. Plugging these values into the above formula for $h'(4)$ gives that $h'(4) = (-2)(9) + (-1)(6) = -24$.

(b) Find $h'(4)$ if $h(x) = g(f(x))$.

We know $h'(x) = g'(f(x))f'(x)$. So, $h'(4) = g'(f(4))f'(4)$. From the table, we can see that $f(4) = 9$ and $f'(4) = 6$. However, we can not determine $g'(9)$ from the given graph of g. So there is MISSING INFORMATION–$g'(9)$.

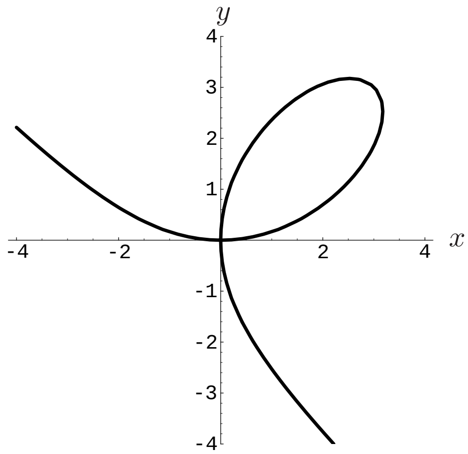
(c) Find $h'(-2)$ if $h(x) = 4\sin(g(x)) - \pi$.

We know that $h'(x) = 4\cos(g(x))g'(x)$. So, $h'(-2) = 4\cos(g(-2))g'(-2)$. From the graph of g, $g(-2) = 0$ and $g'(-2) = \frac{-2}{3}$ (compute the slope of the line through $x = -2$). Plugging in these values into the above equation gives $h'(-2) = 4(-2/3) = -8/3$.

(d) Find $h'(1)$ if $h(x) = (g(x))^2$.

We know that $h'(x) = 2g(x)g'(x)$. So, $h'(1) = 2g(1)g'(1)$. Notice though that on the graph of g, there is a sharp point at $x = 1$. This means that $g'(1)$ is undefined, and thus finding $h'(1)$ is NOT POSSIBLE.

4. (12 points) An example of Descartes' folium, shown in the picture below, is given by $x^3 + y^3 = 6xy$.



(a) Show that the point (3,3) is on the graph.

To show that the point (3,3) is on the graph, we must check that plugging in $x = 3$ and $y = 3$ into the given equation makes both the left hand side and the right hand side equal to each other. Indeed, $x^3 + y^3 = 3^3 + 3^3 = 54$ and $6xy = 6(3)(3) = 54$, so the point (3,3) is on the graph.

(b) Find the equation of the tangent to the graph at the point $(3, 3)$. Show your work.

We must first implicitly differentiate the given equation:
$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$
$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$
$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$
$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$.
So, at the point (3,3), the slope of the tangent to the curve equals $\frac{6(3)-3(3^2)}{3(3^2)-6(3)} = \frac{18-27}{27-18} = -1$. The equation of the tangent line is then $y - 3 = -(x - 3)$ or $y = -x + 6$.

(c) For what value(s) of x (if any) will the tangent to this curve be horizontal? [You do not need to solve for both x and y–just show x in terms of y.] Show your work.

A tangent to the curve will be horizontal if $\frac{dy}{dx} = 0$. This will only happen if the numerator in the equation found for $\frac{dy}{dx}$ in part (b) equals zero. So, we set $6y - 3x^2 = 0$ and solve for x. This gives:
$3(2y - x^2) = 0$
$2y - x^2 = 0$
$2y = x^2$. So, $x = \pm\sqrt{2y}$.

5. (14 points) A family of functions is given by $r(x) = \dfrac{a}{x}e^{bx}$ for a, b, and $x > 0$.

(a) For what values of a and b does the graph of r have a local minimum at the point $(4, 5)$? Show your work and **all supporting evidence** that your function satisfies the given properties.

We begin by computing $r'(x)$:
By the product rule, $r'(x) = \frac{ab}{x}e^{bx} + \frac{-a}{x^2}e^{bx}$
$= \frac{a}{x}e^{bx}(b - \frac{1}{x})$.
Now, the local minimum of r will occur when the derivative is equal to zero. So we set $r'(4)$ equal to zero: $0 = \frac{a}{4}e^{4b}(b - \frac{1}{4})$. Since $\frac{a}{4}e^{4b} > 0$ for all values of $a > 0$ and $b > 0$, we can just set $(b - \frac{1}{4}) = 0$. Doing so yields $b = \frac{1}{4}$.
Now, we must find a. We are given that the point $(4, 5)$ is on the graph of r. So we plug in $x = 4$ and $b = \frac{1}{4}$ into the equation for r and set it equal to 5:
$5 = \frac{a}{4}e^{(\frac{1}{4})4}$.
Solve for a to get that $a = \frac{20}{e}$.
We must now check to make sure that these values of a and b do indeed make the graph of r have a local minimum at the point $(4, 5)$. To check this, we compute the second derivative of r, plug in our values for a and b, and then plug in $x = 4$. This will tell us the concavity of the graph of r at $(4, 5)$, which will in turn, tell us whether we have a local maximum or a local minimum.
$r''(x) = \frac{\frac{20}{e}}{x}e^{\frac{1}{4}x}(\frac{1}{4} - \frac{1}{x})(\frac{1}{4} - \frac{1}{x}) + \frac{\frac{20}{e}}{x}e^{\frac{1}{4}x}(\frac{1}{x^2})$
$= \frac{20}{ex}e^{\frac{1}{4}x}((\frac{1}{4} - \frac{1}{x})^2 + \frac{1}{x^2})$.
Since $\frac{20}{ex}e^{\frac{1}{4}x} > 0$ for $x > 0$, it suffices to look at the sign of $((\frac{1}{4} - x)^2 + \frac{1}{x^2})$. Plugging in $x = 4$ here gives $(\frac{1}{4} - \frac{1}{4})^2 + \frac{1}{4^2} > 0$. Thus we have that the graph of r is concave up at $x = 4$ and our values of a and b do indeed make r have a local minimum at $x = 4$.

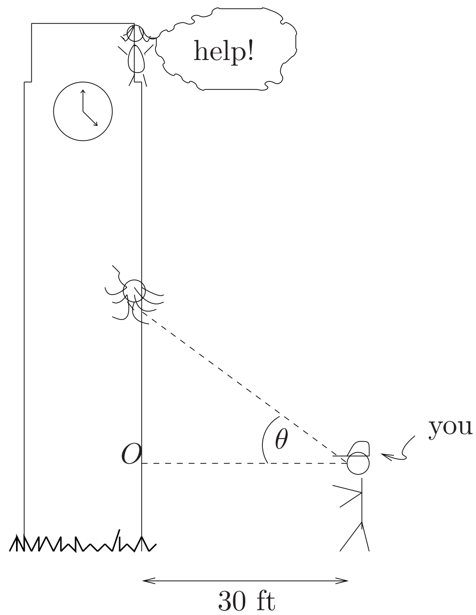
(b) Write an explicit formula for $r(x)$. Circle your answer.

From (a), we know that $r(x) = \frac{20}{ex}e^{\frac{x}{4}} = \frac{20}{x}e^{(\frac{x}{4}-1)}$.

(c) Is the graph of r concave up or down for $x > 0$? Explain using arguments based on calculus–not only from a graph.

From (a), we know that $r''(x) = \frac{a}{x}e^{bx}((b - \frac{1}{x})^2 + \frac{1}{x^2})$. Since $a > 0$, $b > 0$, and $x > 0$, we know that $\frac{a}{x}e^{bx} > 0$. So it suffices to check the sign of $((b - \frac{1}{x})^2 + \frac{1}{x^2}) + \frac{1}{x^2}$. $(b - \frac{1}{x})^2 > 0$ for all $x > 0$ and $\frac{1}{x^2} > 0$ for all $x > 0$. So putting these pieces together, we see that $r''(x) > 0$ for all $x > 0$ and thus the graph of r is concave up for $x > 0$.

6. (10 points) Dr. Octopus is holding Mary Jane Parker hostage at the top of Burton Tower. Spiderman decides to climb the bell tower to try and rescue Mary Jane. Suppose you are standing 30 ft. away from the base of the tower watching Spiderman as he climbs. Let θ be the angle between the line of your horizon and your line of sight to Spiderman. The picture below may help you. [Picture not to scale.]



(a) Find a formula for the rate of change of Spiderman's distance from the point O with respect to θ.

Let x be Spiderman's distance from the point O. So a formula for the rate of change of Spiderman's distance from the point O will be given by $\frac{dx}{d\theta}$. Notice that $\tan(\theta) = \frac{x}{30}$. So, $\frac{dx}{d\theta} = \frac{30}{\cos^2\theta}$.

(b) If the distance from point O to Mary Jane is 200 ft. and Spiderman is climbing at a constant 8 ft/sec, what is the rate of change of θ with respect to time when Spiderman reaches Mary Jane?

We will use the formula found in (a) to solve this part. We're given that $\frac{dx}{dt} = 8$ ft/sec and we are trying to determine $\frac{d\theta}{dt}$. So we must figure out the value of θ when Spiderman is at the top of the bell tower. We do this by solving for θ in the equation:
$\tan(\theta) = \frac{200}{30}$. So, $\theta = \tan^{-1}(\frac{20}{3})$. Plugging these values into the equation from (a) gives:

$$8 = \frac{30}{(\cos(\tan^{-1}(\frac{20}{3})))^2}\frac{d\theta}{dt}.$$

So, $\frac{d\theta}{dt} \approx 0.0059$ radians/sec.

7. (10 points) The University has made an agreement with the Student Government Association to sell more student season football tickets. The tickets will cost $150 each for the first 20,000 tickets. After 20,000 have sold, students will sign up for tickets. For each additional student (over 20,000) that signs up, the season price will be reduced by $0.01 (yes, one cent) per student. A maximum of 35,000 total student tickets will be set aside. Students may sign up for tickets until August 20th. Effective on August 21st, the additional students may pick up their tickets at the reduced rate that has been determined by the number of students who had signed up by August 20th.

(a) What total number of student sales maximizes the university's revenue from student season football tickets? Show your work. [For full credit, you must show the function(s) you use for this problem. Just plugging numbers into a table will not suffice. In addition, show evidence of the use of calculus to find your answer–not merely a graph. State clearly what any variables in your function(s) represent.]

We know that the university's revenue will be given by the price per ticket times the number of tickets sold. Notice that for the first $20,000$ tickets, the price per ticket is fixed. Let t be the number of tickets sold, and let R be the university's revenue from student season football tickets. So for $0 \le t \le 20,000$, $R(t) = 150t$. Let's now consider what happens after $20,000$ students purchase a ticket. We need to calculate the price per ticket. For each additional student (over $20,000$), the price will be reduced by $0.01. The number of additional students is $t - 20,000$, so the total price reduction (just for the additional students) is given by $0.01(t - 20,000)$. Therefore, the price per ticket (for the additional students) is $150 - 0.01(t - 20,000)$. So the university's revenue for the additional tickets is $(150 - 0.01(t - 20,000))(t - 20,000)$. Their total revenue is the revenue from the first $20,000$ tickets sold plus the revenue from the additional tickets sold: $(150 - 0.01(t - 20,000))(t - 20,000) + 150(20,000)$. Thus,

$$R(t) = \begin{cases} 150t & 0 \le t \le 20,000 \\ (150 - 0.01(t - 20,000))(t - 20,000) + 150(20,000) & 20,000 < t \le 35,000 \end{cases}.$$

To find the maximum, we will analyze the two parts of the function $R(t)$ separately. Notice that the first part ($0 \le t \le 20,000$) is a linear function with positive slope, so its maximum occurs at $t = 20,000$. The maximum value is $20,000(150) = 3$ million dollars. To calculate the maximum of the other portion of the function, we take the derivative of $R(t) = (150 - 0.01(t - 20,000))(t - 20000) + (150)(20,000) = -0.01t^2 + 550t - 4,000,000$:

$R'(t) = -0.02t + 550$. Setting this equal to zero yields $t = 27,500$. We can see that this is a maximum since $R''(t) = -0.02 < 0$ for all values of $t \ge 20,000$. Notice that $R(27,500) = 3,562,500$ dollars. Since this is greater than $R(20,000)$, the maximum occurs at $t = 27,500$.

(b) What is the maximum revenue from student season ticket sales (based on this problem)?

From the solution above to (a), this is $\$3,562,500$.

(c) How many additional (i.e., over 20,000) season tickets would be optimal financially from the students' point of view? (Consider here only the students over the initial 20,000.) Explain.

From the work done in (a), the cost of an additional student's ticket is given by the function $c(t) = 150 - 0.01(t - 20,000) = -0.01t + 350$. We want to minimize this function. Notice that we are dealing with a linear function, so the minimum will occur when t is greatest, ie. when $t = 35,000$. We can see that when $t = 35,000$, $c(35,000) = 0$ and thus the additional students would get free tickets. From the students' point of view $15,000$ additional season tickets would be optimal.

8. (20 points) For each of the following, circle *all* correct answers. In each case, there may be more than one item which is correct.

(a) The function f' is continuous everywhere and changes from negative to positive at $x = a$. Which of the following *must* be true?

- a is a critical point of f.
- $f(a)$ is a local maximum of f.
- $f(a)$ is a local minimum of f.
- $f'(a)$ is a local maximum.
- $f'(a)$ is a local minimum.

(b) A function g is defined on all points of a closed interval. Which of the following *must* be true?

- g must have both a global maximum *and* a global minimum.
- g is differentiable on the interval.
- g has no critical points.
- g is continuous on the interval.
- None of the above statements *must* be true.

(c) For the graph of a cubic polynomial $ax^3 + bx^2 + cx + d$, $(a > 0)$, the signs of $f'(0), f''(0)$ and $f'''(0)$ (respectively) could be which of the following? (Circle all that are possible.)

- $-, 0, +$
- $-, 0, -$
- $+, +, +$
- $-, +, -$
- $+, -, +$

(d) The graph of $y = h(x)$ has a local max at $x = 3$ on the closed interval [0,5]. Which of the following *must* be true?

- $h'(3)$ is equal to zero or $h(3)$ is an end point.
- h has a critical point at $x = 3$.
- $h''(3)$ is positive.
- $h''(3)$ is negative.
- None of the statements *must* be true.

(e) Which of the following ***cannot*** be computed using L'Hopital's rule?

- $\lim_{x\to 0}(\sin x/x)$
- $\lim_{x\to 0}(\cos x/x)$
- $\lim_{x\to 0}(x/\sin x)$
- $\lim_{x\to\infty}(x/e^x)$
- $\lim_{x\to\infty}(\sin x/x)$