MATH 115 — FINAL EXAM

April 25, 2005

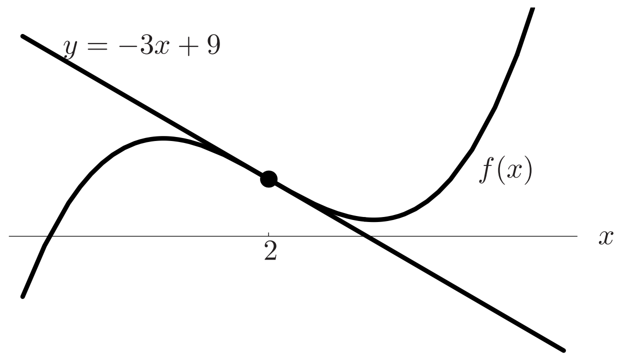
NAME: Solution Key

INSTRUCTOR: ______________________ SECTION NO: __________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of one 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and other sound devices, and remove all headphones.

PROBLEM	POINTS	SCORE
1	12	
2	5	
3	5	
4	10	
5	12	
6	12	
7	17	
8	15	
9	12	
TOTAL	100	

1. (3+3+3+3 points) The figure below shows the tangent line approximation of $f(x)$ near $x = a$.



(a) What are a, $f(a)$, and $f'(a)$?

$a =$ 2 $f(a) =$ 3 $f'(a) =$ -3

(b) Estimate $f(2.1)$. Is this an overestimate or an underestimate? Why?

$f(2.1) \approx$ 2.7 is an underestimate because the tangent line approximation of $f(x)$ for $x > 2$ lies below the graph of $f(x)$.

(c) Estimate $f(1.98)$. Is this an overestimate or an underestimate? Why?

$f(1.98) \approx$ 3.06 is an overestimate because the tangent line approximation of $f(x)$ lies above the graph of $f(x)$ for $x < 2$.

(d) Would you expect your estimation for $f(2.1)$ or $f(1.98)$ to be more accurate? Why?

The tangent line approximation is increasingly more accurate the closer one gets to $x = 2$. Since $2.1 - 2 = 0.1$ and $2 - 1.98 = 0.02$, we would expect $f(1.98)$ to be more accurate.

2. (5 points) Suppose $\int_4^9 (4f(x)+7)dx = 315$. Find $\int_4^9 f(x)dx$.

$$\int_4^9 4f(x)dx + \int_4^9 7dx = 315$$
$$4\int_4^9 f(x)dx + 35 = 315$$
$$4\int_4^9 f(x)dx = 280$$
$$\int_4^9 f(x)dx = 70$$

3. (5 points) Use the Fundamental Theorem to determine the positive value of b if the area under the graph of $f(x) = 4x + 1$ between $x = 2$ and $x = b$ is equal to 11.

$$\int_2^b (4x+1)dx = 11$$
$$\frac{4x^2}{2}|_2^b + x|_2^b = 11$$
$$(2b^2 - 8) + (b - 2) = 11$$
$$2b^2 + b - 21 = 0$$
$$(2b+7)(b-3) = 0$$
$$b = \frac{-7}{2}, 3$$

Since b is positive, $b = 3$.

4. (2 points each–no partial credit) Suppose $\int_a^b f(x)dx = 2$ and $\int_a^b g(x)dx = 6$. Evaluate the following expressions, if possible. If the expression cannot be evaluated with what is given, simply indicate "Insufficient information." Assume that all functions are continuous on the interval $[a, b]$.

(a) $\int_a^b (g(x))^2 dx - (\int_a^b g(x)dx)^2$

Insufficient information: we are not given the value of $\int_a^b (g(x))^2 dx$.

(b) $\int_a^a (h(x))dx$

This integral is equal to 0, since the integral of any function from a point to itself is zero.

(c) $\int_{a+2}^{b+2} f(x-2)dx$

This integral is equal to 2 since it is equal to $\int_a^b f(x)dx$.

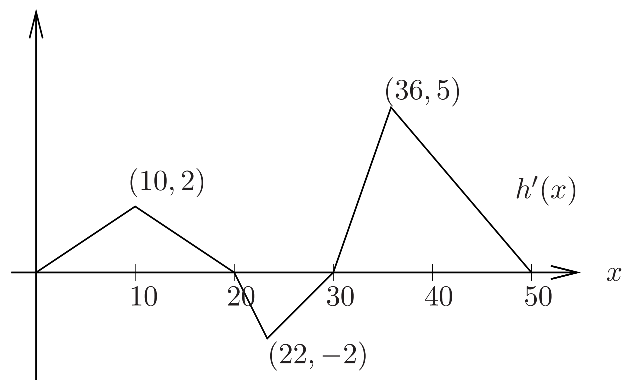
(d) $\int_a^b (f(x)g(x))dx$

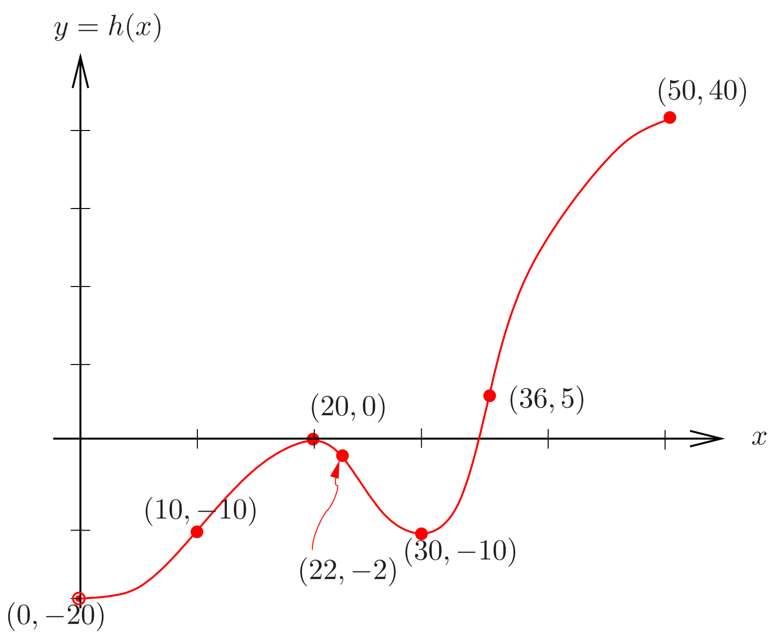
Insufficient information. We do not know how to evaluate the integral of a product of two arbitrary functions.

(e) $\int_b^a (g(x))dx$

This equals $-\int_a^b g(x)dx = -6$.

5 . (12 points) Using the graph of h' in the figure below and the fact that $h(0) = -20$, sketch the graph of $h(x)$. Give the coordinates of all critical points, inflection points, and end points of h. Pay attention to the concavity of the graph.





The points $(10, -10), (22, -2)$ and $(36, 5)$ are inflection points. The graph of $h(x)$ has a local and global min at $(0, -20)$, a local min at $(30, -10)$, a local max at $(20, 0)$, and a local and global max at $(50, 40)$.

6. (4+4+4 points) Harry Potter, Ron, and Hermione decide to attend the Wizard Fair. The newest ride at the fair, called **The Coil of Doom**^{TM}, is a spin-off on bungee jumping. Riders are attached to a special bungee cord which oscillates up and down. The riders' position above the ground, in feet, is given as a function of time, t, in seconds, by $y = y_0 \cos(\omega t) + C$, with y_0, ω, and C constants.

(a) The riders board from a platform 15 feet above the ground, are pulled upward until, 6 seconds later, they reach a maximum height of 165 feet. In another 6 seconds, riders are back at the initial position. The cycle repeats for one minute, at which point the ride ends. Using this information, determine an explicit formula for y. [Show all constants in *exact* form.]

The riders start from a platform 15 feet above the ground and reach a maximum height of 165 ft. The midline is $C = 90$ and the amplitude must be $\dfrac{165-15}{2} = 75$ feet. Since the ride starts at the bottom, , $y_0 = -75$ The period is the time it takes the riders to return to their original position. So, the period equals 12 seconds. Since $\omega = \dfrac{2\pi}{\text{period}}$, $\omega = \dfrac{\pi}{6}$. This means that $y = -75\cos(\dfrac{\pi}{6}t) + 90$.

(b) Find formulas for the velocity and acceleration of the riders as a function of t.

$v(t) = y' = 75(\dfrac{\pi}{6})\sin(\dfrac{\pi}{6}t)$

$a(t) = y'' = 75(\dfrac{\pi}{6})^2\cos(\dfrac{\pi}{6}t)$

(c) Show that the function y satisfies the equation $\frac{d^2y}{dt^2} + \omega^2 y = K$, where K is a constant. What is the value of K?

$\dfrac{d^2y}{dt^2} + \omega^2 y = a(t) + \omega^2 y$ and from part (a) we know $\omega = \frac{\pi}{6}$

$$= 75(\frac{\pi}{6})^2\cos(\frac{\pi}{6}t) - (\frac{\pi}{6})^2((75)cos(\frac{\pi}{6}) + 90)$$

$$= 75(\frac{\pi}{6})^2\cos(\frac{\pi}{6}t) - (\frac{\pi}{6})^2(75)cos(\frac{\pi}{6}) + 90(\frac{\pi}{6})^2$$

$$= \frac{90\pi^2}{36} = \frac{5\pi^2}{2}$$

So $K = \dfrac{5\pi^2}{2}$.

7. (17 points) At the Wizard Fair, there is a booth where wizards win Bertie Bott's Every Flavor Beans. To determine how many beans one gets, a contestant is given a string 50 inches long. From this string, contestants can cut lengths to form an *equilateral triangle* and a *rectangle whose length is twice its width*. The number of Bertie Bott's beans one wins depends on the combined areas of the triangle and rectangle. Harry, knowing calculus, goes immediately to work setting up a function, finding critical points, etc.

(a) Use your knowledge of calculus to determine the areas of the triangle and rectangle that will maximize the number of beans that Harry can win. Show your work.

We need to find a formula for the total area of the triangle and rectangle. Let's begin by finding a formula for the area of the triangle. Say we cut the string x inches from the left end of the string, and suppose the left piece is used to make the equilateral triangle and the right piece is used to make the rectangle. So, each side of the triangle must have length $\frac{x}{3}$. An equilateral triangle has all of its angles equal to $\frac{\pi}{3}$. Let h be the height if the triangle. Then $sin(\frac{\pi}{3}) = \frac{h}{\frac{x}{3}}$ and $h = \frac{x\sqrt{3}}{6}$. So area of triangle $= (\frac{1}{2})(\frac{x}{3})(\frac{x\sqrt{3}}{6})$.

Now we need to find a formula for the area of the rectangle. Let the two sides of the rectangle be l and w. Then the perimeter of the rectangle must equal the length of the right piece of string. So, $2l + 2w = 50 - x$. We know though that the length is twice the width, which means $2(2w) + 2w = 6w = 50 - x$, and therefore, $w = \frac{50 - x}{6}$. Since the area of the rectangle equals lw, we know that the area equals $2w^2 = 2\frac{(50-x)^2}{36}$.

$$\text{Thus we have } A_{total} = \frac{x^2\sqrt{3}}{36} + \frac{2(50-x)^2}{36}$$

$$= \frac{1}{36}((\sqrt{3}+2)^2 - 200x + 2(50)^2).$$

So $\frac{dA}{dt} = \frac{1}{36}(2(\sqrt{3}+2)x - 200)$. Setting the derivative equal to zero to find the critical point gives $0 = (\sqrt{3}+2)x - 100$ and thus $x \approx 26.79$. Notice though that $\frac{d^2A}{dt^2} = \frac{1}{18}(\sqrt{3}+2) > 0$, which means that A has a *minimum* at $x = 26.79$ and the maximum must occur at one of the endpoints - ie. when $x = 0$ or $x = 50$.

$x = 0$: $2w + 2l = 6w = 50 \Rightarrow w = 8.34$. Thus, $A_{total} = lw = 2(8.34)^2 = 136.89$ square inches.

$x = 50$: area $= \frac{1}{2}(\frac{50}{3})(\frac{50\sqrt{3}}{6}) \approx 120.28$ square inches.

This analysis says that Harry should not cut the string and the maximum occurs when the area of the triangle is zero and the area of the rectangle is 136.89 square inches.

(b) If the number of beans won is 9 times the combined area, what is the greatest number of beans a contestant can win?

This is just nine times the area we found in (a). So, the greatest number of beans a contestant can win is $9 * 136.89 = 1232$ beans.

8. (3 points each) Harry, Ron, and Hermione are all thrilled about their abundance of Bertie Bott's Every Flavor Beans; however, they prefer Chocolate Frogs to Bertie Bott's Beans. Luckily, at the wizard fair there is a booth where wizards are able to exchange Bertie Bott's Beans for Chocolate Frogs. The number of beans, N, needed to "purchase" F chocolate frogs is given by the function $N = C(F)$. Using *complete* sentences, give the practical interpretations of each of the following statements in the context of this problem.

(a) $C(3)$

$C(3)$ is the number of beans needed to purchase 3 chocolate frogs.

(b) $C'(3) = 18$

If you purchase 3 chocolate frogs, you need approximately 18 more beans to purchase an additional frog.

(c) $C^{-1}(91)$

$C^{-1}(91)$ is the number of chocolate frogs you'll receive for exchanging 91 beans.

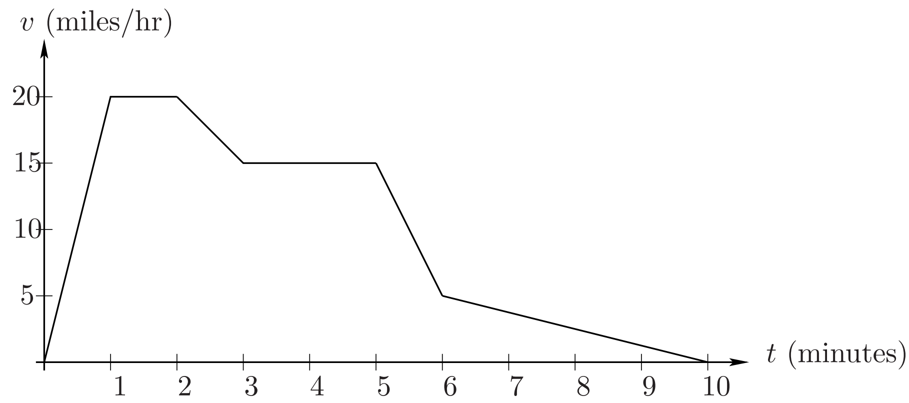
(d) $(C^{-1})'(91) = 0.05$

If you exchange 91 beans for chocolate frogs, you'll get approximately $\frac{1}{20}$ of a chocolate frog by exchanging one more bean.

(e) $\int_4^{10}(C'(F))dF$

The expression $\displaystyle\int_4^{10}(C'(F))dF$ represents the difference between the number of beans needed to purchase 10 chocolate frogs and the number of beans needed to purchase 4 chocolate frogs–i.e., the additional beans needed to go from 4 to 10 frogs.

9. (5+2+2+3 points) The three happy wizards leave the fair and go home to watch the Simpsons. In this episode, Homer needs to deliver Lisa's homework to her at school, and he must do so before Principal Skinner arrives. Suppose Homer starts from the Simpson home in his car and travels with velocity given by the figure below. Suppose that Principal Skinner passes the Simpson home on his bicycle 2 minutes after Homer has left, following him to the school. Principal Skinner is able to sail through all the traffic and travels with constant velocity 10 miles per hour.



(a) How far does Homer travel during the 10 minutes shown in the graph?.

To calculate how far Homer travels during the 10 minutes shown in the graph we find the area under the graph of Homer's velocity. Note that we must multiply by a constant so that the units are correct! This gives that distance $\dfrac{1}{60}$(area under curve)$= \frac{1}{60}(97.5) = 1.625$ miles.

(b) What is the average of Homer's velocity during the 10 minute drive?

average $= (\dfrac{1}{10}) \displaystyle\int_0^{10} v(t)dt = 9.75$ miles/hr.

(c) At what time, $t > 0$, is Homer the greatest distance ahead of Principal Skinner?

As long as Homer's velocity is greater than Principal Skinner's velocity, Homer is becoming farther away from Principal Skinner. Since Principal Skinner is traveling at a constant velocity of 10 miles/hr, Homer is the greatest distance ahead of Skinner at $t \approx 5.5$ minutes.

(d) Does Principal Skinner overtake Homer, and if so, when? Explain.

Principal Skinner will overtake Homer when the distance he has traveled is equal to the distance that Homer has traveled. Notice though that the area under Homer's velocity curve and the area under Principal Skinner's velocity curve overlap. So, they will have traveled the same distance when the area between Homer's velocity curve and Skinner's velocity curve from $t = 0$ to $t = 5.5$ equals the area between the two velocity curves from $t = 5.5$ to some time $t > 5.5$. Notice though that the area between the two curves from $t = 0$ to $= 5.5$ is greater than the area between the two curves from $t = 5.5$ to $t = 10$. So, Skinner does not overtake Homer. Or, more precisely, Principal Skinner travels $\frac{1}{60}(10)(8) = 1.33$ miles. This is less than the 1.625 miles that Homer traveled in the 10 minutes.