Math 115 – First Midterm Exam

Solutions

Name: ____________________

Instructor: ____________________ Section Number: ____________________

1. **Do not open this exam until you are told to begin.**
2. This exam has 8 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

Problem	Points	Score
1	18	
2	16	
3	6	
4	9	
5	14	
6	11	
7	11	
8	15	
Total	100	

1. (18 points) The first and only edition of a certain calculus book came out in 1994. A partial publisher's record reflecting the price p, in dollars, of the book t years after it was first published is given below.

t	0	2	4	6	8
p	67	61	54	46	37

(Assume p and its derivative are differentiable functions.)

(a) (5 pts.) Use the data to estimate $p'(6)$. Show your work; include units.

• $p'(6) \simeq \left(\frac{37-46}{2} + \frac{46-54}{2}\right)\frac{1}{2} = \left(\frac{-9}{2} + (-4)\right)\frac{1}{2} = -4.25$ dollars per year.

(b) (4 pts.) Use your answers from part (a) to give a practical interpretation of $p'(6)$. You should only use everyday language that a non-calculus student would understand.

In 2000 the price of the book was dropping by about 4 dollars and 25 cents per year.

(c) (3 pts.) You somehow find out that $p'(8) = -5.25$. What is the most reasonable estimate of the price of the book in 2003? Show brief work.

• $p(9) \simeq 37 - 5.25 = 31.75$ dollars.

So the book was about 31 dollars and 75 cents in 2003.

(d) Just below the table given above the publisher has scribbled "$p''(t) > 0$."

(i) (3 pts.) Based on the table's data, is it likely that the publisher's scribbled assertion is correct? *Please circle* Yes *or* No *below* ***and*** *briefly explain.*

Yes $\quad$ $\boxed{\text{No}}$

According to the table's date the price function is concave down.

(ii) (3 pts.) Assuming the publisher is correct, what would the publisher's assertion tell a *non calculus expert* about the price of the book during the 8 years following its publication? *Please circle your choice.*

(A) The function p was concave up during the 8 year period following the book's publication.

(B) The function p decreased at an increasing rate during the 8 years that followed the book's publication.

(C) The book was cheapest sometime around 1999.

$\boxed{\text{(D)}}$ The book's price dropped fast at first, then slower and slower toward the end of the 8 year-period.

2. (16 points) In each case below find a possible formula for the function described.

   (a) (4 pts.) Upon looking at her watch a student leaves behind a freshly made cappuccino in a study hall. The coffee is initially 92° F and cools at a rate of 2% per hour. The student knows that, if left there to cool forever, the cappuccino will eventually approach the temperature of the study hall which is 67° F.

   - $y = (92 - 67)(0.98)^t + 67$, or $y = 25(0.98)^t + 67$

   (b) (4 pts.) A sinusoidal function that fits the following table of values:

x	0	1.5	3	4.5	6
$s(x)$	100	300	500	300	100

   - $y = -200\cos\left(\frac{\pi}{3}x\right) + 300$

   (c) (3 pts.) The length, L in feet, of a scarf you are knitting is a *linear function* of the number of rolls, r, of yarn used to knit the scarf, and you know $L'(3) = 0.5$.

   - $L(r) = 0.5r$

   (d) (5 pts.) This rational function has only two zeros, $x = -2$ and $x = 3$. It has only one vertical asymptote at $x = 0$ and a horizontal asymptote of $y = 4$.

   - $y = \dfrac{4\ (x+2)(x-3)}{x^2}$

3. (6 points) Write the **limit definition** for the derivative of $\log(x^2+2)$ with respect to x. (There is no need to simplify or to attempt to find the limit.)

If $f(x) = \log(x^2+2)$, then $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

So in this case the derivative is,

- $\lim_{h \to 0} \dfrac{\log((x+h)^2+2) - \log(x^2+2)}{h}$

4. (9 points) Consider the function $y = j(x)$ graphed below.

Fill in the blanks with all the labelled x values (if any) on the graph satisfying each of the specified conditions. If there are no values which satisfy the condition, write "none."

- The function j is discontinuous here: $g,\ h$
- The function j is not differentiable here: $b,\ d,\ g,\ h$
- The function j' is zero here: none
- The function j' is negative here: e
- The function j'' is positive here: a

5. (2 points each) Circle "TRUE" or "FALSE" for each of the following problems. Circle "TRUE" only if the statement is *always* true. No explanation is necessary.

(a) If $g(x)$ is an everywhere differentiable function, then so is $f(x) = ag(x - h) + b$, where a, b and h are constants.

TRUE　　FALSE

(b) Suppose $H(t)$ and $T(t)$ are differentiable functions, and $T(t) = H(t) - 4$. Then H and T have the same derivative at each t.

TRUE　　FALSE

(c) If l and m are inverse functions and the graph of m crosses the line $y = x$, the graph of l must also cross this line at the same point.

TRUE　　FALSE

(d) If b is a positive constant, then $\lim_{h\to 0} \frac{\sqrt{b+h} - \sqrt{b}}{h} = 0.5\, b^{-1/2}$.

TRUE　　FALSE

(e) If $s(t)$ gives the position of an object moving at a constant velocity, then the object's instantaneous velocity at $t = a$ is equal to $\frac{s(b) - s(a)}{b - a}$ for all $a \neq b$.

TRUE　　FALSE

(f) If t is a differentiable concave up function, then $t'(a) < \frac{t(b) - t(a)}{b - a}$ for all $a < b$.

TRUE　　FALSE

(g) For any constant a, the equation $ax = e^{2\ln x} + a^2$ has exactly one solution.

TRUE　　FALSE

6. (11 points) A fresco supposedly painted by the Italian Renaissance artist Alessandro Botticelli (1445-1510) currently contains 92% of its carbon-14 (half-life 5730 years.) From this information, decide whether Botticelli could have painted the fresco. Show step-by-step calculations, and briefly explain your conclusion.

∘ $\frac{1}{2}C_0 = C_0\ b^{5730}$, which means: $\left(\frac{1}{2}\right)^{1/5730} = b$, or $b \simeq 0.999879$

∘ $0.92\ C_0 = C_0\ 0.999879^t$, which means: $\frac{\ln(0.92)}{\ln(0.999879)} = t$, or $t \simeq 689.29$ years.

So, about 689.29 years have passed since the fresco was done, or the painting was done either in the year 1316 or 1317 (since $2006 - 689.29 = 1316.71$.)

This is *before* Botticelli was born, so he could have not painted the fresco.

7. (11 points) Suppose $P = m(t)$ is the population of Mexico in millions, where t is the number of years since 1980. Explain the meaning of the statements below. You should only use terms and phrases understandable to someone who has never taken calculus. (Assume that the population function is invertible.)

   (a) (3 pts.) $\left.\dfrac{dP}{dt}\right|_{t=0} > 0$

   - In 1980, Mexico's population was increasing.

   (b) (4 pts.) $m^{-1}(97.5) = 18$

   - In 1998 (18 years after 1980) Mexico's population was 97.5 million people.

   (c) (4 pts.) $(m^{-1})'(97.5) = 0.46$

   - When 97.5 million people lived in Mexico, it took about half a year for the population to increase by another million people.

8. (15 points) The functions $r = f(t)$ and $V = g(r)$ give the radius and the volume of a commercial hot air balloon that is being inflated for testing. The variables t and r are measured in minutes and feet respectively, while the volume V is measured in cubic feet. The inflation begins at $t = 0$.

   In each case, translate the words or phrases given below into the precise mathematical expression that represents them. This mathematical expression will consist *only* of numbers, variables and symbols defined in this problem, and other mathematical symbols related to function notation and operations. For example, in problem 1(b) the notation "$p'(6)$" is a mathematical expression. There we gave an expression and you were asked to interpret. Here we are giving the interpretation and you are to supply the mathematical expression.

   You may assume V and r are strictly increasing, differentiable functions.

   (a) The average rate of change in the volume of the balloon when the radius expands from 10 to 12 feet:

   Mathematical Expression $(g(12) - g(10))/2$

   (b) The volume of the balloon t minutes after inflation began:

   Mathematical Expression $g(f(t))$

   (c) The volume of the balloon if the radius was twice as big:

   Mathematical Expression $g(2r)$

   (d) The time elapsed when the radius of the balloon is 30 feet:

   Mathematical Expression $f^{-1}(30)$

   (e) The time elapsed when the volume of the balloon is 10,000 cubic feet:

   Mathematical Expression $f^{-1}(g^{-1}(10,000))$