Math 115 – Second Midterm Exam

Solutions

Name: ______________________________

Instructor: ____________________ Section Number: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

Problem	Points	Score
1	20	
2	10	
3	18	
4	6	
5	10	
6	12	
7	16	
8	8	
Total	100	

1. (20 points) A small company called Maple, Inc. is designing a fancy gift box with a square base. The box must have a volume of 3000 cm^3. The gift box has a lid which is to be made of a material that costs \$1 per square centimeter. The material for the sides of the box costs \$0.75 per cm^2, and the material for the bottom is \$0.80 per cm^2.

   (a) (10 pts.) What are the dimensions of the cheapest gift box the company can make?

   Let x stand for the side of the square base/lid, h stand for the height of the box, and C stand for the cost of making one gift box.

   We then have:

   - $x^2h = 3000$;
   - $(0.80 + 1)x^2 + 4(0.75)xh = C$;

   Solving for $h = 3000/x^2$ in the first equation, and substituting into the second, we obtain the following cost equation:

   $$C = 1.8x^2 + +9000/x.$$

   Solving for x in $C'(x) = (3.6x^3 - 9000)/x^2 = 0$, we obtain $x = 2500^{1/3} \simeq 13.57$ cm. So, for this value of x, $h = 3000/2500^{2/3} \simeq 16.29$ cm.

   Since

   $$C''(x) = 3.6 + 18000/x^3 > 0 \text{ for all } x > 0,$$

   we see that:

   - $C''(2500^{1/3}) > 0,$ and C has a local minimum at $x = 2500^{1/3}$,
   - This local minimum is a global minimum since $C(x)$ is concave up for $x > 0$.

   So, the **dimensions of the cheapest gift box** are approximately

   $$x \simeq 13.57 \text{ cm and } h \simeq 16.29 \text{ cm},$$

   where h is the height of the box, and x is the length of the side of the square base.

   It turns out that Maple, Inc. also produces a cube-shaped wooden box to store jewelry. The cost of producing q of these boxes is given by

   $$C(q) = 8600 + 0.0001(q - 80)^3(q + 90).$$

   (b) (3 pts.) What is the marginal cost when 80 boxes are made? Show your work.

   Marginal cost when 80 boxes are made: $C'(80)$

   Since,

   - $C'(q) = 0.0003(q - 80)^2(q + 90) + 0.0001(q - 80)^3$, and
   - $C'(80) = 0$;

   then the marginal cost when 80 boxes are made is *zero dollars per box.*

(This problem continues on the next page.)

(This is a continuation of Problem 1).

(c) (3 pts.) The marginal cost of producing 95 of the cube-shaped jewelry boxes is about \$13 per box. Explain what this means in practical terms. (Your explanation should be understandable to someone who does not know calculus or economics language).

The cost of producing 96 boxes is about \$13 more than the cost of producing 95 boxes.

(d) (4 pts.) Let R and P denote, respectively, the revenue and the profit of Maple, Inc. from selling q of the cube-shaped jewelry boxes. Fill in the blank and circle the right choice in the paragraph below, as indicated.

If the profit P is maximized when 95 jewelry boxes are sold, then

$R'(95) =$ 13 ________ dollars per box *(fill in the blank)*, and $P''(95)$ must be

POSITIVE / NEGATIVE / ZERO *(circle the appropriate choice).*

2. (10 points) Suppose f has a continuous derivative whose values are given in the following table.

x	0	1	2	3	4	5	6	7	8	9	10
$f'(x)$	5	2	1	-2	-5	-3	-1	2	3	1	-1

(a) Using the data in the table, estimate x-coordinates of indicated critical points of f for $0 < x < 10$.

Since f' is continuous, f' is never undefined and the only critical points are values of x for which $f'(x) = 0$. From the table above, we see that f' changes sign between $x = 2$ and $x = 3$, between $x = 6$ and $x = 7$, and between $x = 9$ and $x = 10$. So, we estimate f has 3 critical points at:

- $x \simeq 2.5$,
- $x \simeq 6.5$,
- $x \simeq 9.5$.

(b) For each critical point above, indicate if it is a local maximum of f, a local minimum, or neither.

- $x \simeq 2.5$ is a local maximum.
  (Since f' changes sign from positive to negative as one moves from left to right in a small neighborhood about $x \simeq 2.5$).
- $x \simeq 6.5$ is a local minimum.
- $x \simeq 9.5$ is a local maximum.

(c) Approximate interval(s) between $x = 0$ and $x = 10$, if any, for which the data indicates that the graph of f is concave up?

The function f is concave up wherever f' is increasing (or where f'' is positive). Looking at the table, we see that f' is increasing approximately for

$$4.5 \leq x \leq 8.5.$$

So, we estimate f is concave up when $4.5 \leq x \leq 8.5$.

(d) If $f(0) = 4$, approximate the value of $f(0.2)$.

The best linear approximation for f at the point (0,4) is given by $y = f'(0)x + b = 5x + b$. Substituting $x = 0$, $y = 4$ in this linear equation, we find $b = 4$.

Therefore,

$$f(x) \simeq 5x + 4 \textbf{ near } x = 0.$$

So, $f(0.2) \simeq 5(0.2) + 4 = 5$, is the approximate value of $f(0.2)$.

3. (18 points)

(a) (2 pts) If $f(x) = ax^4 - x^3 + d$ $(a \neq 0)$ and f has a global maximum, what must be the sign of a? Explain.

The sign of a must be **negative** for f to have a global maximum. If a were positive, then $f \to \pm\infty$ as $x \to \pm\infty$, eliminating the possibility of a global maximum.

(b) (4 pts) Determine all critical points of f.

- $f'(x) = 4ax^3 - 3x^2 = x^2(4ax - 3)$,
- f is a polynomial, so the only critical points are those x for which $f'(x) = 0$,
- $f'(x) = 0 \Rightarrow x = 0$ or $x = 3/(4a)$.

Hence, the critical points are $x = 0$ and $x = 3/(4a)$, $a < 0$.

(c) (4 pts) For what value of x does the maximum occur? Show your work.

Since

$$f''(x) = 12ax^2 - 6x = x(12ax - 6),$$

- $f''(3/(4a)) = 9/(4a) < 0$ (as $a < 0$), so there is a local maximum at $x = 3/(4a)$;
- $f''(0) = 0$, so the second derivative test does not work for this critical point. Note that for $x > 3/(4a)$, f' is negative, so $x = 0$ is neither a max or a min. Since the end behavior of f is toward $-\infty$ as $x \to \pm\infty$, and $x = 3/(4a)$ is the only local max, it is the global max.

(d) (4 pts) For what value(s) of x (if any) does f have inflection points?

Since

$$f''(x) = 12ax^2 - 6x = x(12ax - 6) = 0 \quad \text{for } x = 0 \text{ or } x = 1/(2a),$$

these are the two possible inflection points. We test the sign of f'' to the left and the right of each of these x values. Since $a < 0$, we have $1/(2a) < 0$, also,

$$f'' < 0 \quad \text{for } x < 1/(2a),$$

$$f'' > 0 \quad \text{for } 1/(2a) < x < 0,$$

and

$$f'' < 0 \quad \text{for } x > 0.$$

This means $x = 0$ and $x = 1/(2a)$ are both inflection points, as the sign of f'' changes around those points.

(e) (4 pts) If $f(0) = 4$ and f has a critical point at $x = -\dfrac{1}{4}$, determine a formula for $f(x)$.

- $f(0) = 4 \Rightarrow d = 4$;
- Since the only critical points are $x = 0$ and $x = 3/(4a)$, we must have: $3/(4a) = -1/4$ or $a = -3$.

So this means,

$$f(x) = -3x^4 - x^3 + 4.$$

4. (6 points) Find the *exact* equation of the linear approximation to the curve $f(x) = 10e^{0.4x}$ having slope equal to 2.

- We want x so that $f'(x) = 4e^{0.4x} = 2$;
- Solving, we find that $x = \ln(0.5)/0.4$;
- Now, $f(\ln(0.5)/0.4) = 5$;
- So, the linear approximation we want is of the form

$$f(x) \simeq 5 + 2(x - \ln(0.5)/0.4).$$

5. (10 points) Find the *exact* coordinates of the point (x, y) where the tangent line to the graph of

$$y^3 - xy = -6$$

is vertical. You should start by differentiating the equation above implicitly with respect to x. Show step-by-step work.

Differentiating implicitly with respect to x we get,

$$\begin{aligned} 3y^2y' - [y + xy'] &= 0 \\ y'(3y^2 - x) - y &= 0 \\ y' &= \frac{y}{3y^2 - x}. \end{aligned}$$

The last expression for y' is undefined if $3y^2 - x = 0$ or $x = 3y^2$. We substitute this expression for x in the original equation to get:

$$\begin{aligned} y^3 - 3y^3 &= -6 \\ y &= 3^{1/3}. \end{aligned}$$

This means that $x = 3y^2 = 3^{5/3}$ when $y = 3^{1/3}$, and so

$$(x, y) = (3^{5/3}, 3^{1/3})$$

are the exact coordinates of the point we want.

6. (12 points) The functions $r = f(t)$ and $V = g(r)$ give the radius and the volume of a commercial hot air balloon that is being inflated for testing. The variables t and r are measured in minutes and feet respectively, while the volume V is measured in cubic feet. The inflation begins at $t = 0$.

Use the information on the tables below to answer questions (i)-(iii). (Question (iv) is independent of the tables.)

t	$f(t)$	$f'(t)$
0	c	d
30	b	x
60	a	z

r	$g(r)$	$g'(r)$
a	b	x
b	c	z
d	x	y

(i) (2 pts.) How fast is the radius of the balloon increasing initially?

d ______ ft/min.

(ii) (2 pts.) Assuming f is always increasing for $0 < t < 60$, how much time has ellapsed (since inflation began) when the radius is growing by z ft/min?

60 ______ minutes.

(iii) (3 pts.) How fast is the volume of the balloon increasing a half hour after inflation began?

zx ______ ft^3/min.

(iv) (5 pts.) *(This item is independent of the previous ones).* It turns out that the balloon's surface area increases with the radius by the formula

$$S = h(r) = 4\pi r^2$$

If the radius of the balloon increases linearly from 5 feet at a rate of 1.5 feet per minute, how fast is the balloon's surface area growing an hour after inflation began? Show your work.

Note first that,

$$r = f(t) = 1.5t + 5.$$

Then,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=60} h(f(t)) &= \left(\left.\frac{dh}{dr}\right|_{r=f(60)}\right)\left(\left.\frac{df}{dt}\right|_{t=60}\right)\\ &= h'(f(60))f'(60)\\ &= h'(95)f'(60) = h'(95)(1.5)\\ &= 8\pi(95)(1.5) = 1140\pi \text{ ft}^2/\text{min}.\end{aligned}$$

So the surface area of the balloon is growing at a rate of 1140π square feet per minute one hour after inflation began.

7. (2 points each) Circle "TRUE" or "FALSE" for each of the following problems. Circle "TRUE" only if the statement is *always* true. No explanation is necessary.

(a) If $f(x)$ is increasing, then $f'(x)$ is increasing.

TRUE **FALSE**

(b) Suppose $f'(a) \geq f'(b)$ whenever $a \leq b$. Then f has no points of inflection.

TRUE FALSE

(c) If $f(x)$ is defined for all x, then $f'(x)$ is defined for all x.

TRUE **FALSE**

(d) If f and g are functions whose second derivatives are defined, then $(fg)'' = fg'' + f''g$.

TRUE **FALSE**

(e) If the radius of a circle is increasing at a constant rate, then so is the area.

TRUE **FALSE**

(f) If $f(x)$ has an inverse function, then the derivative of the inverse function is $1/f'(x)$.

TRUE **FALSE**

(g) If $f'(1) = -3.4$ and $g'(1) = 4.1$, then the function $h(x) = f(x) + g(x)$ is increasing at $x = 1$.

TRUE FALSE

(h) The graph of $y = xe^{-0.1x}$ has an inflection point at $x = 20$.

TRUE FALSE

8. (8 points) Below are the graphs of two functions f and g and their derivatives. Consider the function h defined by

$$h(x) = (f(x))^2 + (g(x))^2.$$

Find approximate values for $h(2)$ and $h'(2)$. [Show your intermediate calculations as well as your final answers below.]

- $h(2) \approx$ 3.25, [Since: $h(2) = (f(2))^2 + (g(2))^2 \simeq 1^2 + (-1.5)^2 = 3.25$].
- $h'(2) \approx$ 13, [Since: $h'(2) = 2f(2)f'(2) + 2g(2)g'(2) \simeq 2(1)(8) + 2(-1.5)(1) = 13$].