MATH 115 – FINAL EXAM

Solutions

NAME: ____________________

INSTRUCTOR: ____________________ SECTION NUMBER: ____________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	14	
2	12	
3	9	
4	12	
5	12	
6	12	
7	12	
8	17	
TOTAL	100	

1. (14 points) Problems (a), (b) and (c) below are independent of each other.

   (a) (5 pts.) Compute the linear approximation to $g(x) = 3\ln(x^2)$ near $x = 1$.

   - $g'(x) = 3\frac{1}{x^2}(2x) = \frac{6}{x}$;
   - $g'(1) = 6$;
   - $g(1) = 3\ln(1) = 0$.

   So, $g(x) \simeq 6(x-1)$ near $x = 1$.

   (b) (3 pts.) Write the limit definition of the derivative of the function $f(x) = e^x - e^{-x}$ at the point $x = a$. You do *not* need to simplify or attempt to compute the limit.

$$f'(a) = \lim_{h \to 0} \frac{e^{a+h} - e^{-(a+h)} - e^a + e^{-a}}{h}.$$

   (c) (6 pts.) Assuming the following table accurately represents the behavior of the continuous function $s(x)$ over the interval $[0, 12]$, approximate the following:

   [*NOTE*: the values in the table are for $s'(x)$, not $s(x)$].

x	0	3	6	10	12
$s'(x)$	-6	-3	0	1.2	17

   (i) $s''(3) \simeq \frac{1}{2}\left(\frac{0+3}{3} + \frac{-3+6}{3}\right) = (1+1)/2 = 1$

   (ii) All intervals in $[0, 12]$ (if any) over which s is decreasing.

   If s is decreasing, then $s' < 0$. So, s is decreasing over $(0, 6)$.

   (iii) All intervals in $[0, 12]$ (if any) over which s is concave down.

   If s is concave down, then s' is decreasing. Since s' is increasing over all of $[0, 12]$, there are *NO intervals* over which s is concave down.

2. (12 points) Problems (a) and (b) below are independent of each other.

   (a) (7 pts.) In each case, calculate the value of the given integral expression. Where appropriate, you may assume that f is a differentiable function. *Your final answer should* **not** *contain any integral symbols and* **should be simplified** *as much as possible.* You may assume the symbols a, b and c represent constants. *Show your work!*

   (i) $\int_a^b cf'(t)\,dt = c\int_a^b f'(t)\,dt = c(f(b) - f(a)).$

   (ii) $\frac{d}{dt}\left(\int_1^2 f(t)\,dt\right) = \frac{d}{dt}(\text{Constant}) = 0.$

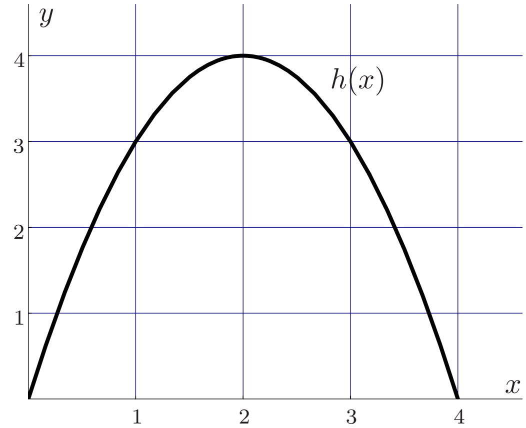
   (iii) $\int_1^3 \left(c + \frac{t^3}{4}\right)\,dt = \left(ct + \frac{t^4}{16}\right)\Big|_1^3 = \left(3c + \frac{81}{16}\right) - \left(c + \frac{1}{16}\right) = 2c + 5.$

   (b) (5 pts.) Assume now that f is a differentiable function of w, and that $w = w(x)$ is a differentiable function of x. Calculate the derivative indicated below. You may assume the symbol a stands for a constant. *Show your work.*

$$\frac{d}{dx}\left(af(w) + xw^2\right) = a\frac{df}{dw}\frac{dw}{dx} + \left(w^2 + 2xw\frac{dw}{dx}\right) = w^2 + \frac{dw}{dx}\left(a\frac{df}{dw} + 2xw\right).$$

3. (9 points) Problems (a) and (b) below are independent of each other.

(a) (6 pts.) The graph of a function $h(x)$ is given below.



- Numbers:

$A = h'(1), \quad B = h'(2), \quad C = h'(3), \quad D = h'(3.001), \quad E = \frac{h(3)}{3}, \quad F = \frac{h(3) - h(2)}{3 - 2}.$

- Write the the numbers A–F from smallest to largest:

$D\ (\simeq -2.2),\ \ C\ (\simeq -2),\ \ F\ (\simeq -1),\ \ B\ (\simeq 0),\ \ E\ (\simeq 1),\ \ A\ (\simeq 2)$

(smallest) (largest)

(b) (3 pts.) Consider the function $w(x)$ given by:

$$w(x) = \begin{cases} -x + 3, & 0 \le x < 1 \\ 2x, & 1 \le x \le 2 \end{cases}$$

Write the numbers L, I, R (defined below) from smallest to largest.

- Numbers:

$L =$ Left-hand sum of w over [0,2] using 2 subdivisions

$I = \int_0^2 w(x)\ dx$

$R =$ Right-hand sum of w over [0,2] using 2 subdivisions.

- Ordered numbers:

$L\ (= 5),\ \ I\ (= 5.5),\ \ R\ (= 6)$

(smallest) (largest)

4. (12 points) Problems (a) and (b) below are independent of each other.

(a) (6 pts.) Suppose the function r gives the number of customers per day going to a new ice-cream store that just opened near campus. (Assume t is measured in days since the opening and that we are modeling the situation by a continuous function, r.) *IMPORTANT: The answers to (i) and (ii) should include clear units, and should be given using words understandable to someone who has never taken calculus.*

(i) What does $\int_0^{20} r(t)\, dt$ represent?

This definite integral represents the total number of *customers* in the ice-cream store during the first 20 days following its opening.

(ii) If each customer spends on average of \$3.50 in the store, what does the following expression represent?

$$\frac{3.5}{20}\int_0^{20} r(t)\, dt$$

This definite integral represents the *average daily revenue* (in *dollars*) of the ice-cream store (from sales to customers) during the first 20 days following its opening.

(b) (6 points) If the average value of the function $d(x) = 7/x^2$ on the interval $[1, c]$ is equal to 1, what is the value of c?

Using the definition of average value we see that:

$$1 = \frac{1}{c-1}\int_1^c \frac{7}{x^2}\, dx, \quad \text{which means that}$$

$$1 = \frac{7}{c-1}\left(\frac{-1}{x}\right)\Bigg|_1^c.$$

Now, the previous equation amounts to:

$$\frac{c-1}{7} = \frac{(1-c)}{c}, \quad \text{or}$$

$$\begin{aligned} c^2 - c &= 7 + 7c, \\ 0 &= c^2 - 8c + 7, \\ 0 &= (c-1)(c-7). \end{aligned}$$

Thus, either $c = 1$ or $c = 7$. Note that $\int_1^1 f(x)d(x) = 0$ for any function, so $c = 7$.

5. (2 points each) Circle "TRUE" or "FALSE" for each of the following problems. Circle "TRUE" only if the statement is *always* true. No explanation is necessary.

(a) $\int (1+y^2)\left(\frac{1}{y}\right)\, dy = (y+y^3)(\ln|y|) + C$

TRUE **FALSE**

(b) If f is any continuous function, then $\int_0^2 f(x)\, dx = \int_0^2 f(t)\, dt$.

TRUE FALSE

(c) If $\int_{-1}^2 g(x)\, dx + 6 = 10$ and g is an odd function, then $\int_1^2 g(x)\, dx = 4$.

TRUE FALSE

(d) If $\int_1^3 f(x)\, dx = 3$, then $\int_1^3 (3f(x)+2)\, dx = 11$.

TRUE **FALSE**

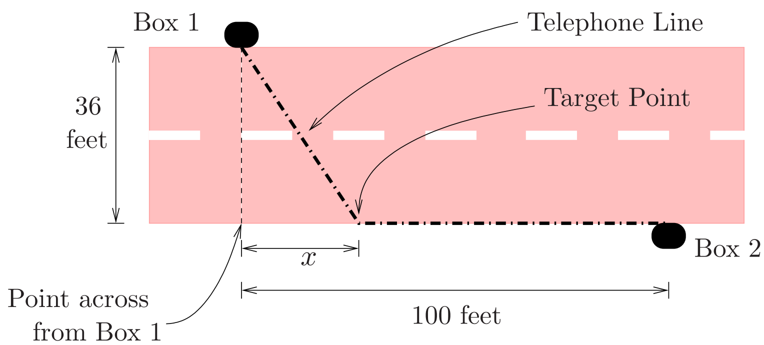
(e) If an object has constant nonzero acceleration, then the position of the object as a function of time is a quadratic polynomial.

TRUE FALSE

(f) If $f''(x)$ is continuous (over all the real numbers) and the graph of f has an inflection point at $x = p$, then $f''(p) = 0$.

TRUE FALSE

6. (12 points) A telephone installation crew must run a line underground between two junction boxes. Unfortunately, there is a 36 feet wide paved road between the two boxes, and one box is 100 feet down that lane from the other (see figure). It costs \$30 per foot to cut and repair the paved road, but only \$24 per foot to dig and refill along the side of the road. The crew will cut and repair the road to a point x feet from the point directly across from the first junction box, and then dig along the road the rest of the way. Determine the number of feet, x, from the point directly across from the first junction box which will minimize the cost of the installation.



Let C stand for the total cost of the installation. Note that C is a function of x. Then:

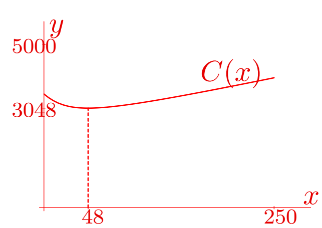
$$\begin{aligned} C(x) &= (\text{Total cost on paved road}) + (\text{Total cost along the road}) \\ &= 30\ (\text{Distance on paved road}) + 24\ (\text{Distance along the road}) \\ &= 30\sqrt{x^2+36^2} + 24(100-x) \\ &= 30\sqrt{x^2+1296} - 24x + 2400, \qquad \textit{and} \qquad C'(x) = \frac{30x - 24\sqrt{x^2+1296}}{\sqrt{x^2+1296}}. \end{aligned}$$

If $C'(x) = 0$, then:

$$\begin{aligned} 30x - 24\sqrt{x^2+1296} &= 0, \qquad \text{which means that} \\ \sqrt{x^2+1296} &= \frac{5x}{4}, \qquad \text{or} \\ x^2 + 1296 &= \frac{25}{16}x^2, \quad \text{or} \qquad x = 48. \end{aligned}$$

So the only candidate value for a minimum of $C(x)$ between 0 and 100, is $x = 48$ feet.

Looking (for instance) at the graph of C against x,



we see that $x = 48$ *is* a local minimum for C in the interval $[0, 100]$. [Note: we could have also used the first or second derivative test to show that $x = 48$ is a local minimum and then evaluated $C(x)$ at $x = 0, x = 48$, and $x = 100$ to show that the minimum occurs at $x = 48$–or, provided an argument that $x = 48$ is the only critical point on the domain of the function and then shown that the minimum occurs there.]

Thus, the crew should dig diagonally across the road to a point 48 feet from the point directly across from the first junction box in order to minimize the cost of the installation.

7. (12 points) You are heading due North across the Mackinac Bridge, and you sight your favorite fudge factory, located 1/4 mile due East of the end of the bridge. You might be fooling yourself, but the minute you sight the factory you are sure you can smell the chocolate fudge. You have to put the car on cruise control (at 55 mph) to resist the temptation to speed the rest of the way across the bridge. Show all work on parts (a) and (b) below.

(a) If you are still 1.25 miles from the end of the bridge when you spot the factory, what is the distance (across the water) between your car and the factory?

- $D = \sqrt{(1.25)^2 + (0.25)^2} \simeq 1.275$ miles.

So, the distance across the water between my car and the factory is about 1.275 miles.

(b) If θ is the angle formed by a line between the factory and end of the bridge and the line from the factory to your car, how fast is θ changing at the time you spot the factory?

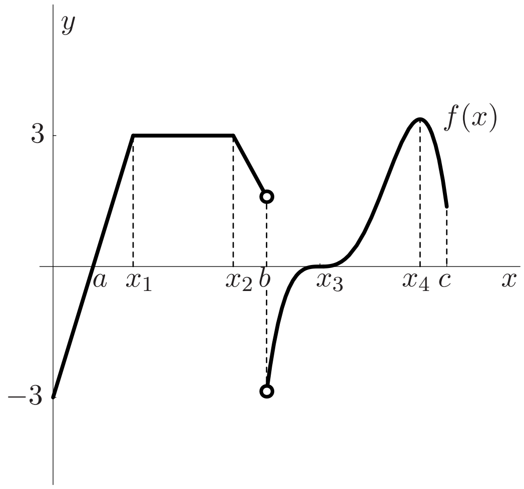
- $\tan\theta = \dfrac{x}{1/4} = 4x$, where
- x is the distance between your car and the end of the bridge, and $\dfrac{dx}{dt} = -55$ mph, so
- $\theta = \arctan(4x)$.

Therefore,

- $\dfrac{d\theta}{dt} = \dfrac{4}{1+(4x)^2}\dfrac{dx}{dt} = \dfrac{4(-55)}{1+16x^2}$, so
- when $x = 1.25$, we have $\dfrac{d\theta}{dt} = \dfrac{-220}{26} = -8.46$ radians/hr.

Thus, the angle θ is decreasing by about 8.46 radians per hour when you first spot the fudge factory!

8. (17 points) Consider the graph of the function f given below. Your answers in parts (i) through (iv) may contain some of the constants a, x_1, x_2, b, x_3, x_4, or c.



(i) (2 pts.) Consider just the interval (b, c). Find all the x-values which are critical points of f on this interval (if any).

Critical points: x_3, x_4 .

(ii) (6 pts.) Determine the following and briefly justify your answers.

- The *value* of $\int_{x_2}^{x_1} f(x)\,dx$: $-3(x_2 - x_1)$.

JUSTIFICATION:

This definite integral is equal to $-\int_{x_1}^{x_2} f(x)\,dx$, and $\int_{x_1}^{x_2} f(x)\,dx$ is equal to the area of the rectangle of height 3 and base $(x_2 - x_1)$ formed by the graph of f, the x-axis, and the lines $x = x_1$ and $x = x_2$.

- The *sign* of $\int_{b}^{c} f(x)\,dx$: positive .

JUSTIFICATION:

The area under the graph of f and above the x-axis, between $x = x_3$ and $x = c$, is *larger* than the area under the x-axis and above the graph of f, between $x = b$ and $x = x_3$.

(iii) (5 pts.) If: $F'(x) = f(x)$ and $F(0) = \pi$, estimate $F(x_2)$. Show step-by-step work.

$$F(x_2) - F(0) = \int_0^{x_2} f(x)\,dx, \quad \text{which means that} \quad F(x_2) = 3(x_2 - x_1) + \pi$$

(iv) (4 pts.) If F (from part (iii)) is a continuous function, determine the x-values of all the critical points of F on the interval $(0, c)$.

Critical points: a, b, x_3 .