MATH 115 – FIRST MIDTERM EXAM

February 6, 2007

NAME: SOLUTIONS

INSTRUCTOR: ______________________ SECTION NUMBER: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	9	
2	12	
3	12	
4	10	
5	8	
6	12	
7	11	
8	12	
9	14	
TOTAL	100	

1. According to a survey by the U-M Transportation Research Institute, gasoline prices are projected to reach \$5.00 a gallon by the year 2020.

(a) (5 points) Assuming that the average gas price in 2007 is \$2.00 per gallon (yes, we know that is wishful thinking), find an exponential function, P, that models the average gas price t years after 2007. Show either an "exact" answer or at least 4 decimal places in your answer.

Since the function we are looking for is exponential, we are looking for a function of the form $P = ab^t$, where a and b are constants, passing through the points $(0, 2)$ and $(13, 5)$.

Using the point $(0, 2)$ we have $0 = ab^0 \Rightarrow a = 2$. Now, using the point $(13, 5)$, combined with what we just showed we have $5 = 2b^{13} \Rightarrow b = (\frac{5}{2})^{\frac{1}{13}}$.

Thus the function we are looking for is $P = 2(\frac{5}{2})^{\frac{t}{13}}$ or approximating to four decimal places $P = 1.0730^t$.

(b) (2 points) What is the *annual* percent change in the average gas price according to this model? (Show to at least one decimal place.)

To find the annual percent change in gas prices we note that the value b we calculated in (a) was approximately 1.0730. Thus the annual percent change in gas prices is (to one decimal place) 7.3%.

(c) (2 points) What is the yearly *continuous* percent rate of change for this model? (Show to two decimal places.)

To find the continuous percent rate of change for the model we have to express $P(t)$ in the form $P(t) = ae^{kt}$, for constants a and k.

$P(t) = 2(\frac{5}{2}^{\frac{1}{13}})^t = ae^{kt} \Rightarrow a = 2$ and $e^k = (\frac{5}{2})^{\frac{1}{13}}$. Solving this for k we get $k = \frac{1}{13}ln(\frac{5}{2}) \approx 0.0705$. Thus, the continuous growth rate is 7.05%.

(d) (5 points) If, instead, gasoline prices grow linearly between 2007 and 2020, find a linear function, L, to model the price t years after 2007.

Since the function we are looking for is linear, we are looking for a function of the form $P = mt + d$, where m and d are constants, passing through the points $(0, 2)$ and $(13, 5)$.

The constant m is the slope, which we can calculate $m = \frac{5-2}{13-0} = \frac{3}{13}$. Since we have the point $(0, 2)$, we have $L = \frac{3}{13}t + 2$.

(e) (2 points) The survey indicates that prices may be \$4.00 per gallon eight years from now. Which of the two models best predicts this projection?

The exponential model predicts an average price of approximately \$3.51 per gallon, and the linear model predicts an average price of approximately \$3.85 per gallon. Thus the linear model is the better model.

2. (9 points) Consider the following equations with a and b constants:

(i) $y^5 = ex$ (ii) $y - a^b = b(x-a)^{1/3}(x-a)^{2/3}$

(iii) $y - 2 = \sqrt{x^e}$ (iv) $\pi y = (9/13)^x$

Use the equations to answer the following. (One equation will not be used.)

(a) Which of the above can be written so that y is a linear function of x?

Equation number __(ii)__

What is the slope of the function? __b__

What is the y-intercept of the graph? __$a^b - ab$__

(b) Which of the above can be written so that y is an exponential function of x?

Equation number __(iv)__

What is the initial value of the function? __$\frac{1}{\pi}$__

What is the percent rate of growth/decay of the function? __$\approx$ -30.8 %__

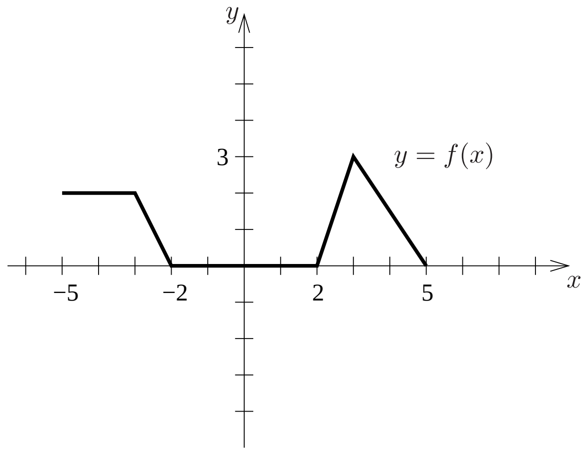
(c) Which of the above equations can be written as a power function of the form $y = kx^p$?

Equation number __(i)__

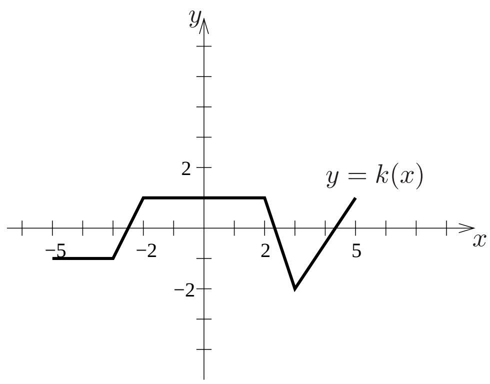
What is k? __$e^{\frac{1}{5}}$__

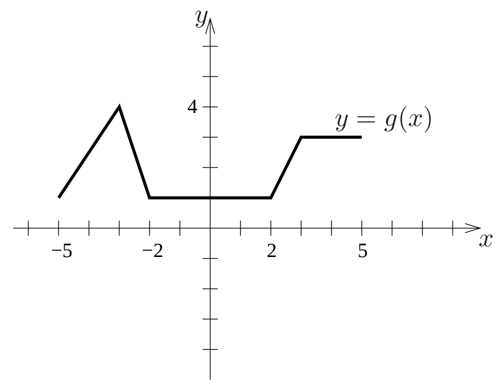
What is p? __$\frac{1}{5}$__

3. (12 points) The graph of $y = f(x)$ is given by the figure below.



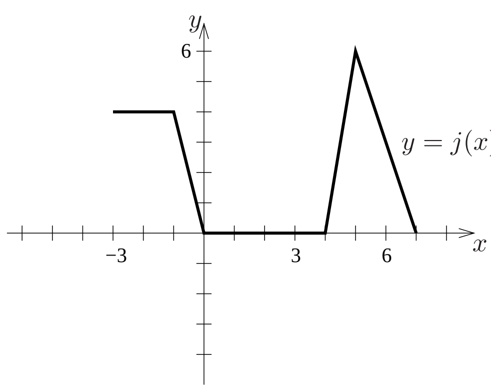
The graphs of the following functions are related to the graph of f. Determine a formula for each graph in terms of the function f.

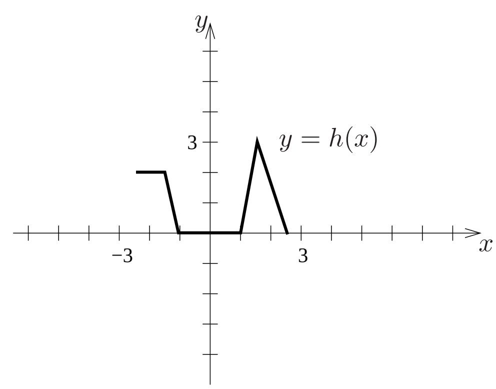




$k(x) =$ $-f(x) + 1$

$g(x) =$ $f(-x) + 1$





$j(x) =$ $2f(x - 2)$

$h(x) =$ $f(2x)$

4. (13 points) Brian's favorite web site is woot.com. This site generally sells one item each day and records the number of sales during each hour of the daily special. On Thursday, Brian noted that the day's graph of sales looked sinusoidal. At 1:00 a.m., there were 70 items sold–and again at 3:00 p.m.. Between those hours, the sales went down (once) to a low of 20 items and then up (once) to a high of 120 items before the last 70 items were sold at 3:00 p.m..

   (a) Determine a trigonometric function that would model sales, S, as a function of t in hours after 1:00 am, assuming that the graph Brian saw was sinusoidal.

$$S(t) = -50\sin(\frac{\pi}{7}t) + 70$$

   (b) What is the period of your function?

The period is ___14 hours___

   (c) What is the amplitude of your function?

The amplitude is ___50___

   (d) Approximately when were the sales increasing fastest?

The sales are increasing the fastest at seven hours past 1:00 am, or at 8:00 am.

5. (12 points) At woot.com the staff has become quite good at predicting the number of items that will be sold based on the brand name, reliability reports, the price, and the predicted popularity of the item. The maximum number of items, N, that they expect to sell during the entire sale period on a given day is a function of what they call the Max Sales Index, i, so $N = f(i)$, where the units of i are referred to as "points."

   (a) In the context of this problem, give a practical interpretation of $f(10)$.

   The expression $f(10)$ represents the maximum number of items that woot.com expects to sell during the entire sale period on a given day when the Max Sales Index is 10 points.

   (b) In the context of this problem, what is the practical interpretation of $f'(5) = 2500$?

   The practical interpretation of $f'(5) = 2500$ is that woot.com expects a product with Max Sales Index of 6 will sell approximately 2500 more items than a product with Max Sales Index of 5.

   (c) The number of Wooters (registered members of Woot.com) is currently over 500,000. Since there is not a mechanism for "un-registering," and the membership has grown very quickly, assume that the number of Wooters, W in thousands, is an invertible function of time, t, in hours, $W = g(t)$. In this context, give a practical interpretation of $(g^{-1})'(200) = .05$

   $(g^{-1})'(200) = 0.05$ means that when the number of Wooters is 200,000, it takes approximately 0.05 hours (or three minutes) for the next 1000 Wooters to register.

   (d) Sometimes woot.com sells bags of junk, "like shopping blindfolded at the Dollar Store." We can't say the exact name here, so we'll call them BoCs. Even these bags sell quickly on woot.com–typically in minutes. A recent BoC sale recorded the following data, where $s(t)$ gives the total number of BoC sales t minutes after the sale began. Use the data to estimate the $s'(10)$. Show your work.

time (minutes)	6	8	10	12	14	16	18
$s(t)$ (number of BoCs)	46	88	136	184	243	313	436

$$s'(10) \approx \frac{s(12) - s(8)}{12 - 8} = \frac{184 - 88}{4} = 24 \text{ BoCs per minute}$$

6. (16 points) State whether each of the following statements are TRUE or FALSE. For each statement, give an explanation. If the statement is false, give an example that shows a contradiction to the statement. If the statement is true, show why it is true. Examples may be formulas or graphs. Explain your reasoning.

(a) If $f'(x)$ is increasing, then $f(x)$ is also increasing.

FALSE
Consider the function $f(x) = x^2$. We know that $f''(x) > 0$ for all x (so that $f'(x)$ is increasing), but $f'(x) = 2x$ is less than 0 for $x < 0$ so that f is decreasing for $x < 0$.

(b) If $f(x) \neq g(x)$ for all x, then $f'(x) \neq g'(x)$.

FALSE
Consider $f(x) = x + 1$ and $g(x) = x + 2$. Then $f'(x) = g'(x) = 1$ even though $f(x) \neq g(x)$.

(c) There is a function which is continuous on [1,5] but not differentiable at $x = 3$.

TRUE
The function $f(x) = |x - 3|$ is one such function.

(d) If a function is increasing on an interval, then it is concave up on that interval.

FALSE
The function $f(x) = \ln x$ is a counterexample.

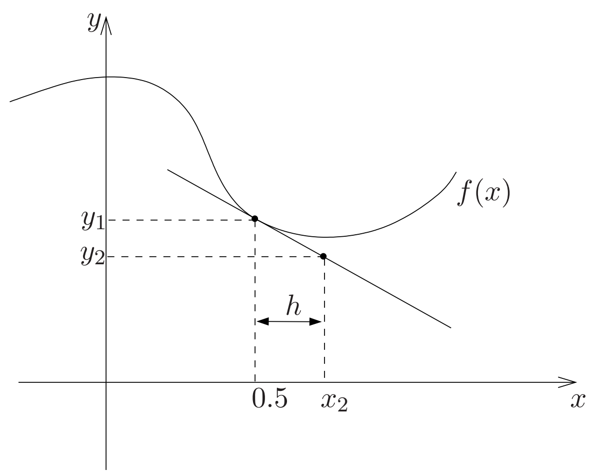
7. (7 points) Use the function

$$g(x) = x^{\sin(x)}$$

to give the *limit definition* for $g'(2)$ [No need to simplify or approximate the limit.]

$$g'(2) = \lim_{h \to 0} \frac{(2+h)^{\sin(2+h)} - 2^{\sin(2)}}{h}$$

8. (7 points) The figure below shows $y = f(x)$ and a line tangent to f at $x = 0.5$. Given that $f(0.5) = 2$, $f'(0.5) = -3$, and $h = 0.1$, determine the values of y_1, y_2, and x_2. [Note: x and y are different scales on the graph.]



$y_1 =$ 2

$y_2 =$ 1.7

$x_2 =$ 0.6

9. (8 points) Cosmologists, through a technique best described as hocus pocus, measure a quantity $T(t)$, the temperature of the universe in degrees Kelvin (K), where t is in gigayears (Gyr) after the Big Bang. Suppose that, currently, $t = 13.6$, $T(13.6) = 2.4$, and $T'(13.6) = -12$.

   [Note: A gigayear is 1 billion years, and the Kelvin temperature scale is an absolute temperature scale where the lowest possible temperature is defined as being zero Kelvin.]

   (a) For each of the following statements, state whether you agree or disagree with the conclusion and justify your reasoning.

      (i) In the next billion years, the temperature of the universe will drop by approximately 12 degrees Kelvin.

      It is not possible for the temperature of the universe to drop by 12 Kelvin in the next billion years because the current temperature is 2.4 Kelvin and the Kelvin temperature scale is absolute (the lowest possible temperature is 0 Kelvin).

      (ii) In the next year, the temperature of the universe will drop by approximately $\frac{12}{1{,}000{,}000{,}000}$ degrees Kelvin.

      It is reasonable to say that the temperature of the universe will drop by approximately $\frac{12}{1{,}000{,}000{,}000}$ Kelvin in the next year since $T'(13.6) = -12$ so as time increases by one billionth of a Gyr from the current time ($t = 13.6$ Gyr) the temperature should decrease by twelve billionths Kelvin.

   (b) Assume $T(t)$ is decreasing and does not change concavity on the domain $[13.6, \infty)$. Do you expect $T(t)$ to be concave up or concave down on the domain $[13.6, \infty)$? Justify your answer using physical reasoning.

   The function $T(t)$ should be concave up since as the universe expands, the temperature of the universe decreases and $T(t)$ should asymptotically approach zero.