Math 115 – Final Exam

April 20, 2007

Name: SOLUTIONS

Instructor: ______________ Section Number: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 10 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

Problem	Points	Score
1	10	
2	12	
3	12	
4	12	
5	16	
6	12	
7	12	
8	8	
9	6	
Total	100	

1. (2 points each, no partial credit) For the following statements circle True or False. Circle True only if the statement is *always* true.

(a) If y is differentiable for all x, then the value of $y'(x)$ is a unique number for each x.

True   False

(b) The only antiderivative of $\cos(x)$ is $\sin(x)$.

True   False

(c) For a continuous function f on the interval $a \leq x \leq b$, if the left-hand sum and the right-hand sum are equal for a given number of subdivisions, then they are equal to $\int_a^b f(x)dx$.

True   False

(d) For the continuous function f, if the units of t are seconds and the units of $f(t)$ are meters, then the units of $\int_0^1 f(t)dt$ are meter seconds.

True   False

(e) For any function f, if $\lim_{x \to 3^-} f(x) = a$ and $\lim_{x \to 3^+} f(x) = a$, then $f(3) = a$.

True   False

2. (12 points) Suppose that f and g are continuous functions and $\int_0^2 f(x)dx = 5$ and $\int_0^2 g(x)dx = 13$. Compute the following. If the computation cannot be made because something is missing, explain clearly what is missing.

(a) $\int_4^6 f(x-4)dx = \int_0^2 f(x)dx = 5$

(b) $\int_{-2}^0 2g(-t)dt = 2\int_0^2 g(t)dt = 2(13) = 26$

(c) $\int_2^0 (f(y)+2)dy = -\int_0^2 f(y)dy - \int_0^2 2dy = -(5) - 4 = -9$

(d) $\int_2^2 g(x)dx = 0$

(e) Suppose that f is an even function. Compute the average value of f from -2 to 2.

$$\frac{1}{2-(-2)}\int_{-2}^2 f(x)dx = \frac{1}{4}\left(2\int_0^2 f(x)dx\right) = \frac{1}{2}(5) = \frac{5}{2}$$

3. (12 points) Octavius, the giant octopus, escaped after all. He was being kept in a temporary tank near the harbor—apparently even less secure than his tank at the zoo. He managed to get into the bay, but the coast guard could keep track of him with a homing device that had been attached to Octavius at the zoo.

(a) The coast guard station is 2 kilometers (2000 meters) down the beach from where Octavius entered the bay. If the octopus was moving directly away from the shore at a constant rate of 25 meters per minute, how fast was the distance between the coast guard and Octavius changing when the octopus was 200 meters from the shore?

Let the distance from Octavius to the shore be given by x, and the distance from Octavius to the Coast Guard station be given by D. Then $D^2 = x^2 + (2000)^2$, and

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt}.$$

We are given that $\frac{dx}{dt} = 25$ m/min, and when $x = 200$ m., $D = \sqrt{200^2 + 2000^2} \approx 2009.98$ m. Thus, $\frac{dD}{dt} \approx 2.49$ m/min when $x = 200$.

(b) At what rate is the angle formed by the beach and the line that gives distance from the coast guard station to Octavius changing when Octavius is 200 meters from the shore?

We know that

$$\tan\theta = \frac{x}{2000},$$

so

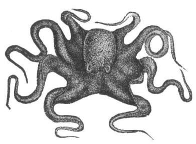
$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{2000}\left(\frac{dx}{dt}\right).$$

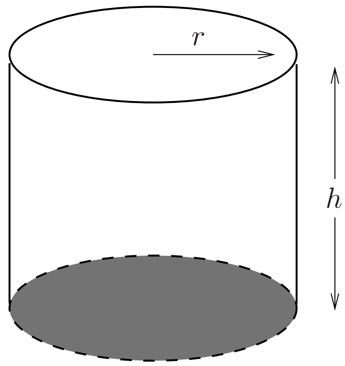
When $x = 200, \cos\theta = \frac{2000}{2010}$.

Thus,

$$\frac{d\theta}{dt} = \frac{(\frac{2000}{2010})^2}{2000}(25) \approx 0.01238 \quad \text{rad/min}.$$

4. (12 points) The zoo has decided to make the new octopus tank spectacular. It will be cylindrical with a round base and top. The sides will be made of Plexiglas which costs \$65.00 per square meter, and the materials for the top and bottom of the tank cost \$50.00 per square meter. If the tank must hold 45 cubic meters of water, what dimensions will minimize the cost, and what is the minimum cost?





Let V denote the volume of the tank. Then $V = \pi r^2 h = 45\ m^3$.

Solving for h gives $h = \dfrac{45}{\pi r^2}$ m.

The area of the "sides" of the tank is $2\pi r h = \dfrac{90}{r}\ m^2$, and the area of the circles for the top and bottom of the tank is $2\pi r^2$.

Thus, the cost of the tank as a function of r is

$$C(r) = 65(\frac{90}{r}) + 50(2)(\pi r^2) = \frac{5850}{r} + 100\pi r^2.$$

To minimize the cost, set $C'(r) = -\dfrac{5850}{r^2} + 200\pi r$ equal to zero, since $C'(r)$ is defined for all values of r in the domain $(r > 0)$.

Solving for r, we get $r^3 = \dfrac{5850}{200\pi}$, so $r \approx 2.104$ meters.

Testing to see if this r value is in fact, the minimum, we use the second derivative test.

Since $C''(r) = 200\pi + 2\dfrac{5850}{r^3} > 0$ for all values of r in the domain, we see that C is concave up for all values of r. Since $C(r) \to \infty$ as $r \to 0$ and as $r \to \infty$, $r = 2.104$ is the global minimum of C.

radius ≈ 2.104 m

height ≈ 3.237 m

cost $\approx \$4171$

5. (16 points) Use the information given in the table below to calculate the indicated values. If a value cannot be determined, state explicitly what is missing. Assume that f and f' are continuous, and that the table is reflective of the behavior of f.

x	0	3	6	9	12
$f(x)$	30	20	13	8	5
$f'(x)$	-4	-3	-2	-1.5	-.5

Determine the following and show your work (3 points each):

(a) an approximate value for $f(3.1)$ using a local linearization

$f(3.1) \approx f(3) + f'(3)(3.1 - 3) \approx 20 + (-3)(0.1) = 19.7$

(b) a left-hand sum with 4 subdivisions to approximate $\int_0^{12} f(x)dx$

$\text{LHS}_{(4)} = (f(0) + f(3) + f(6) + f(9))(3) = 213$

(c) the least number of subdivisions necessary to assure that the left- and right-hand approximations of $\int_0^{12} f(x)dx$ agree to within 1 unit

If $|RHS - LHS| \leq 1$, then $|f(12) - f(0)|\Delta x = 25\Delta x \leq 1$. Thus, $25\left(\frac{12-0}{n}\right) \leq 1 \Rightarrow (25)(12) \leq n$. This implies we need at least 300 subdivisions.

(d) $\int_3^{12} f'(x)dx$

From the FTofC, we know $\int_3^{12} f'(x)dx = f(12) - f(3) = 5 - 20 = -15$

Explain your answers to the following (2 points each):

(e) Do you expect your approximation for $f(3.1)$ from part (a) to be an overestimate or an underestimate?

If the table is representative of the behavior of the function f, then $f''(3) > 0$ which imples that f is concave up at 3. Thus we expect the approximation to be an underestimate.

(f) Do you expect your left-hand approximation from part (b) to be an overestimate or an underestimate?

If the table is representative of the behavior of f, then f is decreasing, thus the left-hand sum is an overestimate.

6. (12 points) Suppose f and g are differentiable functions with values given by the table below. **To receive full credit, for each part below first show the formula for the derivative in terms of x, and *then* find the requested value.**

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	6	9	14	-7
2	4	13	1	-11

(a) If $h(x) = f(x)g(x)$, find $h'(1)$.

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$\Rightarrow h'(1) = f'(1)g(1) + f(1)g'(1) = (14)(9) + (6)(-7) = 84$

(b) If $j(x) = \dfrac{\ln(x)}{f(x)}$, find $j'(1)$

$$j'(x) = \frac{\frac{f(x)}{x} - \ln(x)f'(x)}{(f(x))^2}$$

$$\Rightarrow j'(1) = \frac{\frac{1}{1}f(1) - \ln(1)f'(1)}{(f(1))^2} = \frac{1}{6}$$

(c) If $d(x) = \sin(\cos(f(x)))$, find $d'(1)$.

$d'(x) = \cos(\cos(f(x)))(-\sin(f(x)))f'(x)$

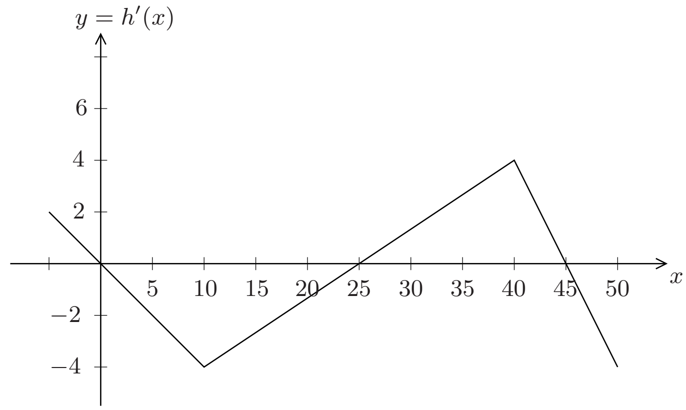
$\Rightarrow d'(1) = \cos(\cos(f(1)))(-\sin(f(1)))f'(1) \approx 2.243$

(d) If $t(x) = g(x)g(2x)$, find $t'(1)$.

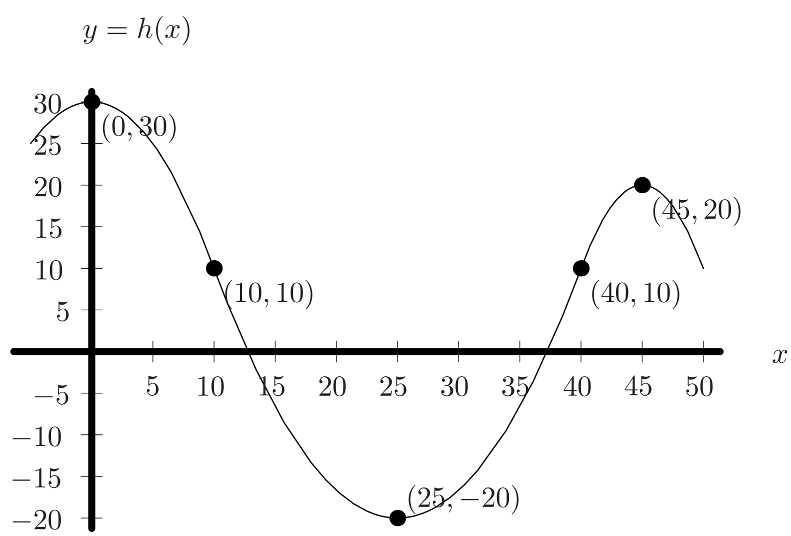
$t'(x) = g'(x)g(2x) + g(x)g'(2x)(2)$

$\Rightarrow t'(1) = g'(1)g(2) + 2(g(1)g'(2)) = -289$

7. (12 points) The graph of the derivative, $h'(x)$, of a function h is given in the figure below. Given $h(10) = 10$, sketch the graph of $h(x)$ on the axes below the figure. Indicate the critical points and inflection points of h *and* give the coordinates of each of those points.



The graph of $h(x)$ is given below. The function h has critical points at (0,30), (25,-20), and (45,20); and h has inflection points at (10,10) and (40,10).



8. (8 points) Consider the family of functions of the form

$$f(x) = x + ax\ln(bx).$$

(a) Under what conditions on a and b is the function f defined and concave down for positive values of x? Explain. [Note: to receive credit, your work must be shown analytically, not just in the form of a graph.]

Since $\ln(y)$ is defined only for $y > 0$, we know that f is defined only for $bx > 0$. Thus we know that $b > 0$.

To ensure that f is concave down we require that the second derivative be negative. Taking the derivative of f we get

$$f'(x) = 1 + a\ln(bx) + ax(\frac{b}{bx}) = 1 + a\ln(bx) + a,$$

so $f''(x) = \dfrac{ab}{bx} = \dfrac{a}{x}$.

Thus we want $\dfrac{a}{x} < 0$, so $a < 0$, since x is positive.

Thus to have f defined and concave down we need $b > 0$ and $a < 0$.

(b) For what value(s) of x (if any), will f have a critical point?

From part (a) we have $f'(x) = 1 + a\ln(bx) + a$. Thus there are no values of x in the domain of f such that f' is not defined.

Solving for values of x such that $f'(x) = 0$ we have

$$\ln(bx) = -\left(\frac{1+a}{a}\right)$$

so

$$bx = e^{-(\frac{1+a}{a})},$$

giving

$$x = \frac{1}{b}e^{-(\frac{1+a}{a})}.$$

9. (6 points)

(a) Write the *limit definition* of $g'(2)$ for an arbitrary differentiable function g.

$$g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}.$$

(b) Use limit and summation notation to define $\int_a^b h(x)dx$ for an arbitrary continuous function h.

$$\int_a^b h(x)dx = \lim_{n \to \infty} \left(\sum_{i=0}^{n-1} h(x_i)\Delta x \right) = \lim_{n \to \infty} \left(\sum_{i=1}^{n} h(x_i)\Delta x \right).$$

[Note: either the LHS or the RHS could be used here. Both are not necessary.]