MATH 115 –FIRST MIDTERM

February 5, 2008

NAME: ****SOLUTIONS**** ______________________________

INSTRUCTOR: ______________________ SECTION NUMBER: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are **??** questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	10	
2	12	
3	14	
4	6	
5	8	
6	14	
7	12	
8	12	
9	12	
TOTAL	100	

1. (2 points each) For each of the following, circle *all* statements which **MUST** be true.

   (a) Let f be a non-decreasing differentiable function defined for all x.

   - $f'(x) \geq 0$ for all x.
   - $f''(x) \geq 0$ for all x.
   - $f(x) = 0$ for some x.

   (b) Let f and g be continuous at $x = -1$, with $f(-1) = 0$ and $g(-1) = 3$.

   - $f \cdot g$ is continuous at $x = -1$.
   - $\dfrac{g}{f}$ is continuous at $x = -1$.
   - $\dfrac{f}{g}$ is continuous at $x = -1$.

   (c) Let f be differentiable at $x = 2$, with $f(2) = 17$.

   - $\lim_{x \to 2} f(x) = 17$.
   - $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = 17$.
   - $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$ exists.

   (d) Let f be defined on $[a, b]$ and differentiable on (a, b), with $f'(x) < 0$ for all x in (a, b).

   - If $a < c < d < b$, then $f(c) > f(d)$.
   - $f''(x) > 0$ for some x in (a, b).
   - f is continuous on (a, b).

   (e) Let f be a twice-differentiable function that is concave-up on (a, b), with $f(a) = 4$ and $f(b) = 1$.

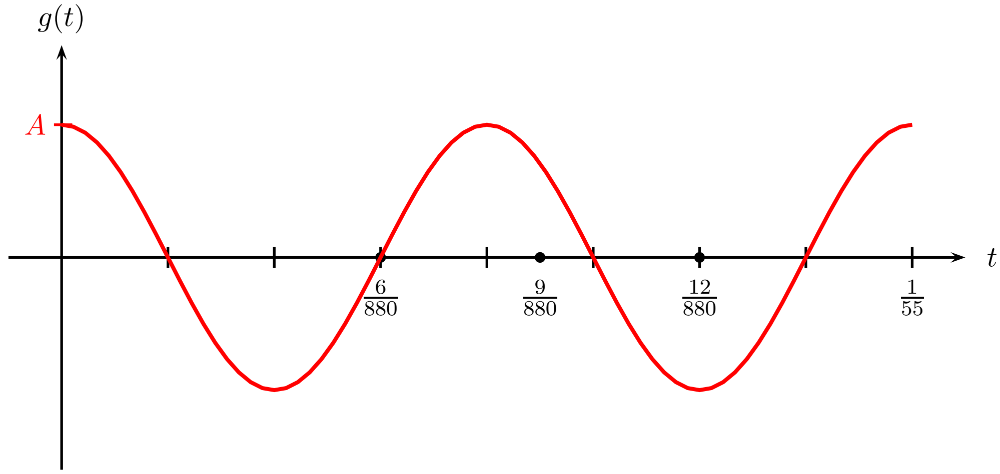
   - For some x in (a, b), $f(x) = 2.5$.
   - For all x in (a, b), $f''(x) \geq 0$.
   - $f'(a) \leq f'(b)$.

2. If you pluck a guitar string, a point P on the string vibrates. The motion of the point P is given by

$$g(t) = A\cos(220\pi\, t),$$

where $g(t)$ is the displacement (in mm) of P from its position before the string was plucked, t is the number of seconds after the string was plucked, and A is a positive constant.

(a) (6 points) Sketch a graph of $g(t)$, for $0 \le t \le 1/55$, on the axes below. Be sure to indicate A on your sketch.



(b) (3 points) Sketch tangent lines to your graph at $t = 6/880$, $t = 9/880$, and $t = 12/880$. Use these to write the numbers $g'(6/880)$, $g'(9/880)$, and $g'(12/880)$ in order from least to greatest.

$$g'(9/880) \quad < \quad g'(12/880) \quad < \quad g'(6/880)$$

(c) (3 points) What is the meaning of A, in terms of the plucked string?

A is the initial displacement (in mm) of P. It is also the maximum displacement of the string.

3. You are driving to Detroit to see a concert at the Majestic Theater. You leave Ann Arbor at 6:00 pm. Let $D(t)$ be your distance from Detroit t minutes after 6:00.

   (a) (3 points) What is the sign (positive or negative) of $D'(t)$, assuming you never turn around on your way to Detroit? Explain.

   $D'(t)$ must be negative, since the function $D(t)$ is a decreasing function: the distance to Detroit decreases as you drive towards the city.

As you approach the city, you notice signs indicating the distance remaining. To pass the time, your friend, riding in the passenger seat, makes the following table:

t, minutes after 6:00	0	3	6	10	15	18	20	22	25	30
$D(t)$, miles from Detroit	50	47	43	38	35	35	32	30	27	20

   (b) (4 points) Use the table to estimate $D'(6)$. Include units.

   Using the difference quotient from the left, we have

$$D'(6) \approx \frac{D(6) - D(3)}{6 - 3} = \frac{43 - 47}{6 - 3} = -\frac{4}{3} \approx -1.33.$$

   From the right, we have

$$D'(6) \approx \frac{D(10) - D(6)}{10 - 6} = \frac{38 - 43}{10 - 6} = -\frac{5}{4} = -1.25.$$

   Averaging the two, we have $D'(6) \approx -\frac{31}{24} \approx -1.29$. In any case, the units are in miles per minute.

   (c) (3 points) Based on your answer to (**??**), approximately what did your speedometer read at 6:06? (Your car's speedometer gives speed in miles/hour.)

   Taking the absolute value of the answer to (**??**), and converting units to miles per hour by multiplying by 60, we see that the speedometer said approximately $60 \cdot \frac{4}{3} = 80$ miles per hour. (Or $60 \cdot \frac{5}{4} = 75$ mi/hr, or $60 \cdot 1.29 = 77.4$ mi/hr, depending on which of the three methods used to solve (**??**).)

   (d) (4 points) Could $D(t)$ be linear ...

   - for $20 \le t \le 30$? Briefly explain.

     No. If it were, its graph would have slope $-1 = \frac{30-32}{22-20}$ for $20 \le t \le 22$, but slope $-\frac{7}{5} = \frac{20-27}{30-25}$ for $25 \le t \le 30$.

   - for $20 \le t \le 25$? Briefly explain.

   This is possible, because the difference quotients $\frac{30-32}{22-20}$, $\frac{37-30}{25-22}$ are both equal to -1.

4. (6 points) A certain state has been setting the date for its primary election using a function $P(x)$, where x is the number of years since 1992 and $P(x)$ is the number of days from the beginning of the year when the primary was held. (Count January 1 as one day from the beginning.) The pattern of elections is given in the table:

x	0	4	8	12	16
$P(x)$	96	48	24	12	6

Assuming that P is either linear or exponential, write a formula for $P(x)$ which accurately reflects the data in the table. If this trend continues, when will the primary be held in 2012? Show your work.

First, P cannot be linear, since $\frac{P(4)-P(0)}{4-0} = \frac{48-96}{4-0} = -12$, but $\frac{P(8)-P(4)}{8-4} = \frac{24-48}{8-4} = -6$. Assuming P is exponential, then, write $P(x) = C \cdot b^x$. Since $P(0) = C$, we have $C = 96$. Since

$$\frac{P(4)}{P(0)} = \frac{C \cdot b^4}{C \cdot b^0} = b^4,$$

we have $b^4 = 48/96 = 1/2$, so $b = \sqrt[4]{1/2} \approx 0.84$. (Note: taking the negative 4th root $b = -0.84$ doesn't make sense in the context of the problem.) Thus

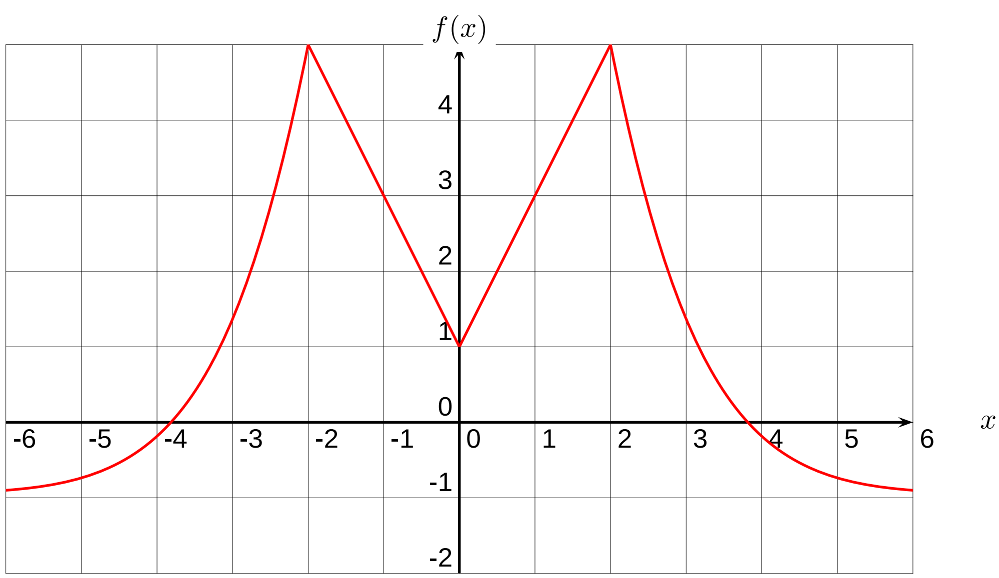
$$P(x) = 96(\sqrt[4]{1/2})^x \approx 96(0.84)^x,$$

and when $x = 20$ (i.e, the year 2012) $P(x) = 3$. The primary will take place on January 3^{rd} in 2012.

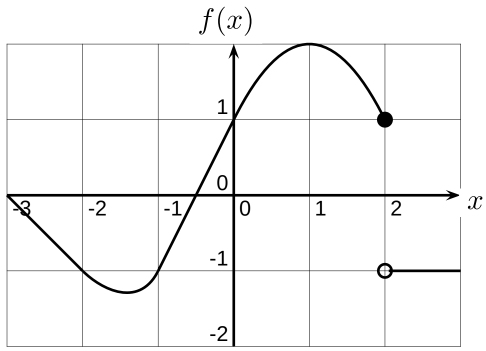
5. (8 points) On the axes below, carefully sketch the graph of a continuous function $f(x)$ with the following properties:

- f is an even function (that is, $f(-x) = f(x)$).
- $f(0) = 1$.
- $f'(x) = -2$ on $(-2, 0)$.
- $f'(x) < 0$ for $x > 2$.
- $f''(x) > 0$ for $x < -2$.
- $\lim_{x\to\infty} f(x) = -1$.

Your graph should be as accurate as possible. (You won't be graded on your draftsmanship, though!)



6. The graph of a function f is shown below, together with a table of values for a function g. Define a third function h by $h(x) = f(x-2)$.



x	$g(x)$
-3	1
-2	0
-1	-1
0	-1
1	2
2	2
3	0

(a) (2 points each) Using the information given, find

i. $f(g(1)) = 1$

ii. $g(h(2)) = 2$

iii. $h(f(0)) = -1$

(b) (3 points) Is it possible that $g = f'$? Briefly justify your answer.

No. There are several reasons: for example, f is not differentiable at $x = 2$, so f' is not defined at 2, but g is defined at 2. Also, from the graph we see that f is increasing on $[-1, 1]$, but g takes negative values there.

(c) (5 points) Is it possible that $g = h'$ on the interval where h is known? Justify.

This is possible. The graph of h is the same as that of f, shifted 2 units to the right. Thus we have information about h on the interval $[-1, 3]$, and the values of g on this interval appear to agree with the slopes of the tangent lines to the graph of h.

7. You've arrived at the Majestic Theater, excited to see your favorite immigrant gypsy punk band play. As soon as the show starts, some people from the audience try to climb onto the stage. Once audience members are on the stage, they refuse to leave the stage. So, the theater has a security team in charge of pushing these people off the stage. Let

   - $P(t)$ be the total number of people on the stage;
   - $B(t)$ be the number of band members on the stage;
   - $S(t)$ be the number of security guards on the stage;
   - $A(t)$ be the number of audience members on the stage;
   - $c(t)$ be the rate at which audience members are climbing onto the stage;
   - $d(t)$ be the rate at which audience members are being pushed off the stage by security;

   all at time t, where t the is number of minutes after the show begins. Since there are so many people at the show, you can assume these are all differentiable functions of t.

   (a) (2 points) What does $P(t) - (B(t) + S(t))$ represent? (Assume everyone in the theater is either in the band, in the audience, or a security guard.)

   $P(t) - (B(t) + S(t)) = A(t)$, the number of audience members on stage at time t.

   (b) (2 points) Write an equation for $A'(t)$ in terms of the appropriate functions from the list above.

   $A'(t) = c(t) - d(t)$.

   (c) (2 points) In the context of this problem, is it possible that $c(t) < 0$?

   No. (It's hard to make sense of a negative rate of people climbing onto the stage, especially since we're told that no one climbs off.)

   (d) (2 points) After one hour, the lead singer tells the guards to stop pushing people off the stage. What does this mean about $d(t)$, for $t > 60$?

   This means $d(t) = 0$ for all $t > 60$.

   (e) (2 points) The show lasts a total of two hours. On the interval $[0, 60]$, is $A(t)$ increasing, decreasing, neither, or is there insufficient information? Explain.

   There is not enough information: the sign of $A'(t) = c(t) - d(t)$ could be positive or negative on the interval $[0, 60]$.

   (f) (2 points) On the interval $[60, 120]$, is $A(t)$ increasing, decreasing, neither, or is there insufficient information? Explain.

   $A(t)$ is non-decreasing on $[60, 120]$. We know $A'(t) = c(t) - d(t) = c(t) \geq 0$ for t in $[60, 120]$, because $d(t) = 0$ on this interval, and $c(t) \geq 0$ always. [Answers here could be "increasing," "non-decreasing," or "insufficient information." The explanation was important in all cases.]

8. Census figures for the US population (in millions) are listed in the table below. Let f be the function such that $P = f(t)$ is the population (in millions) at year t.

Year	1950	1960	1970	1980	1990
Pop.	150.7	179.0	205.0	226.5	248.7

Assume that f is increasing, so f is invertible.

(a) (3 points) What is the meaning of $f^{-1}(200)$?

This is the year when the US population was 200 million.

(b) (3 points) What does the derivative of $f^{-1}(P)$ at $P = 200$ represent? What are its units?

The units of $(f^{-1})'(200)$ are years per millions of people. This number represents the approximate time (in years) for the population of the U.S. to grow from 200 million to 201 million people.

(c) (3 points) Estimate $f^{-1}(200)$.

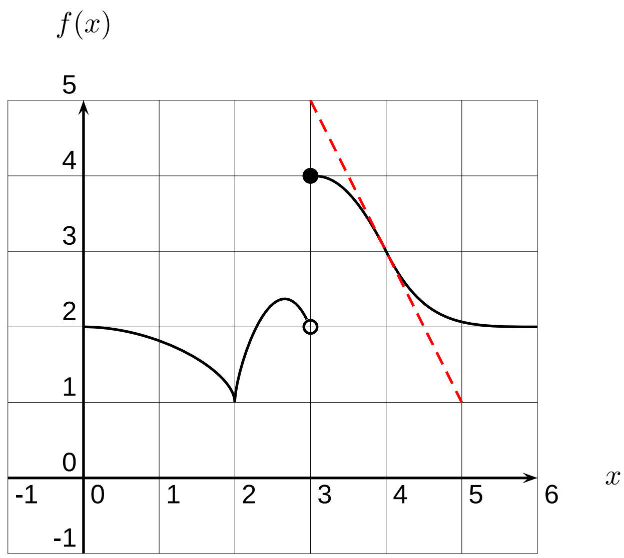
$f^{-1}(200)$ must be between 1960 and 1970. If the increase is linear, the population during that period is increasing by 2.6 million people per year. Reasonable estimates are 1968 or 1967, because the rate of growth appears to be slowing down. [Again, answers may vary, but work and reasons were important.]

(d) (3 points) Estimate the derivative of $f^{-1}(P)$ at $P = 200$.

Using the estimate from (**??**) and the difference quotient from the right, we have

$$(f^{-1})'(200) \approx \frac{f^{-1}(205.0) - f^{-1}(200)}{205.0 - 200} = \frac{1970 - 1968}{205 - 200} = \frac{2}{5} = 0.4.$$

9. A function f is defined on the interval $[0, 6]$. The graph of $y = f(x)$ is shown below.



(a) (2 points) On which intervals does it appear that f is continuous?

Judging from the graph, f appears to be continuous on $[0, 3)$ and on $(3, 6]$. It is not continuous at $x = 3$, because the value jumps there.

(b) (3 points) On which intervals does it appear that f is differentiable?

It appears that f is differentiable on $(0, 2)$, $(2, 3)$, and $(3, 6)$. It cannot be differentiable at $x = 3$, because it is not continuous there. At $x = 2$, there is a sharp point, suggesting that f is not differentiable there, either.

(c) (3 points) Does $\lim_{x\to 3} f(x)$ exist? If so, estimate it; if not, explain why.

$\lim_{x\to 3} f(x)$ does not exist, because $f(x)$ approaches 2 as x approaches 3 from the left, but $f(x)$ approaches 4 as x approaches 3 from the right. Since these are different values, the limit cannot exist.

(d) (4 points) Estimate $f'(4)$ and find an equation of the tangent line to the graph of f at $x = 4$.

Drawing the tangent line, we it has slope -2, so $f'(4) = -2$. Using the point $(4, 3)$, we can write the equation of the tangent line as

$$(y - 3) = -2(x - 4),$$

or equivalently,

$$y = -2x + 11.$$