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MATH 115 – SECOND MIDTERM

March 25, 2008

NAME: ______________________ SOLUTIONS ______________________

INSTRUCTOR: ______________________ SECTION NUMBER: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 8 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	10	
2	11	
3	10	
4	15	
5	16	
6	12	
7	16	
8	10	
TOTAL	100	

1. (2 points each) For each of the following, circle all the statements which are **always** true. For the cases below, one statement may be true, **or** both **or** neither of the statements may be true.

   (a) Let $x = c$ be an inflection point of f. Assume f' is defined at c.

   - If L is the linear approximation to f near c, then $L(x) > f(x)$ for $x > c$.
   - The tangent line to the graph of f at $x = c$ is above the graph on one side of c and below the graph on the other side.

   (b) The differentiable function g has a critical point at $x = a$.

   - If $g''(a) > 0$, then a is a local minimum.
   - If a is a local maximum, then $g''(a) < 0$.

   (c) The derivative of $g(x) = (e^x + \cos x)^2$ is

   - $g'(x) = 2\left(e^x - \sin x\right)\left(e^x + \cos x\right)$.
   - $g'(x) = 2e^{2x} + 2\left(e^x \cos x - e^x \sin x\right)$.

   (d) A continuous function f is defined on the closed interval $[a, b]$.

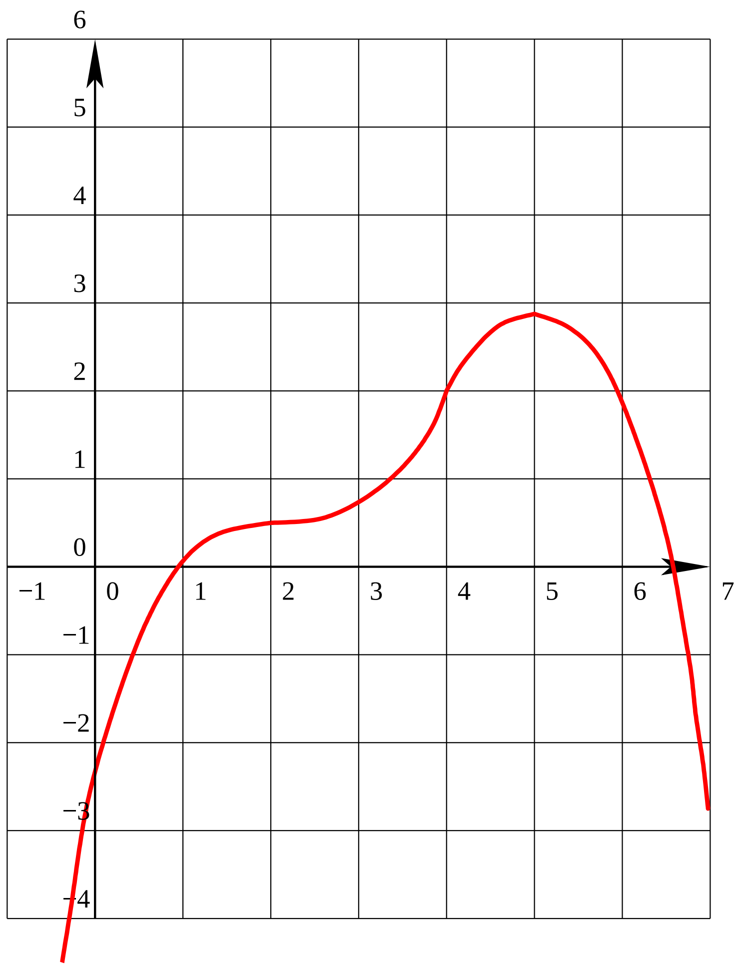
   - f has a global maximum on $[a, b]$.
   - f has a global minimum on $[a, b]$.

   (e) Consider the family of functions $e^{-(x-a)^2}$.

   - Every function in this family has a critical point at $x = 0$.
   - Some function in this family has a local maximum at $x = 2$.

2. (11 points) Let $f(x)$ be a **continuous** function defined for all real numbers x. Sketch a possible graph of f, given that

- $f(4) = 2$;
- $f'(x) > 0$ **and** $f''(x) < 0$ for $x < 2$;
- $f'(2) = 0$ and $f''(2) = 0$;
- $f''(x) > 0$ for $2 < x < 4$;
- $f''(4) = 0$;
- $f''(x) < 0$ for $x > 4$;
- $f'(x) > 0$ for $2 < x < 5$;
- $f'(5) = 0$;
- $f'(x) < 0$ for $x > 5$.



3. (10 points) Suppose $f'(x)$ is a differentiable increasing function for all x. In each of the following pairs, circle the larger value. In each case, give a **brief** reason for your choice.(Assume that none of the values below are equal for this function and $\Delta x \neq 0$).

(a) $f'(5)$ and $\boxed{f'(6)}$

f' is increasing, and $5 < 6$.

(b) $\boxed{f''(5)}$ and 0

f' is increasing, so its derivative f'' is always positive (since $f''(5) \neq 0$) .

(c) $\boxed{f(5+\Delta x)}$ and $f(5) + f'(5)\Delta x$

Since $f''(5) > 0$, f is concave up at 5. Therefore its values near 5 are greater than the linear approximation, which is $f(5) + f'(5)\Delta x$.

4. (15 points) Using calculus, find constants a and b in the function $f(x) = axe^{bx}$ such that $f(\frac{1}{3}) = 1$ and the function has a local maximum at $x = \frac{1}{3}$. Once you have found a and b, verify that your answer satisfies the given conditions. Show all work.

Since $f(\frac{1}{3}) = 1$, we have an equation $1 = a\left(\frac{1}{3}\right)e^{b/3}$, so solving for a gives

$$a = 3e^{-b/3}.$$

Now for f to have a local maximum at $\frac{1}{3}$, we want $f'(\frac{1}{3}) = 0$ and $f''(\frac{1}{3}) < 0$. Let's solve the first equation, using our expression for a.

$$\begin{aligned} f'(x) &= ae^{bx} + abxe^{bx} \\ &= 3e^{-b/3}e^{bx} + 3bxe^{-b/3}e^{bx}, \\ f'\left(\frac{1}{3}\right) &= 3e^{-b/3}e^{b/3} + be^{-b/3}e^{b/3} \\ &= 3 + b, \end{aligned}$$

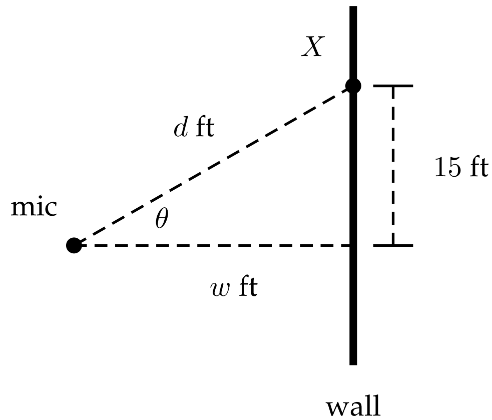
and setting this equal to zero tells us $b = -3$. Plugging this into the expression for a, we get $a = 3e$.

Here, verify that $f(\frac{1}{3}) = 1$ and $f'(\frac{1}{3}) = 0$. However, most important, check the critical point to see that it is a local maximum. We use the second derivative test, and check $f''(\frac{1}{3}) < 0$:

$$\begin{aligned} f''(x) &= -9ee^{-3x} - 9ee^{-3x} + 27exe^{-3x} \\ &= -18ee^{-3x} + 27exe^{-3x}; \\ f''\left(\frac{1}{3}\right) &= -18ee^{-1} + 27e\left(\frac{1}{3}\right)e^{-1} \\ &= -18 + 9 \\ &= -9 < 0. \end{aligned}$$

Thus, the critical point at $x = \frac{1}{3}$ is a local maximum.

5. (16 points) A directional microphone is mounted on a stand facing a wall. The sensitivity S of the microphone to sounds at point X on the wall is inversely proportional to the square of the distance d from the point X to the mic, and directly proportional to the cosine of the angle θ. That is, $S = K\frac{\cos\theta}{d^2}$ for some constant K. (See the diagram below.) How far from the wall should the mic be placed to maximize sensitivity to sounds at X?



We are given that $S = K\frac{\cos\theta}{d^2}$. Also, from the definition of cosine we see $\cos\theta = \frac{w}{d}$. By the Pythagorean theorem, $d^2 = w^2 + 15^2 = w^2 + 225$. Therefore

$$S = K\frac{\cos\theta}{d^2} = K\frac{\frac{w}{d}}{w^2+225} = K\frac{w}{(w^2+225)^{3/2}}.$$

Differentiating, we get

$$\frac{dS}{dw} = K\frac{(w^2+225)^{3/2} - 3w^2(w^2+225)^{1/2}}{(w^2+225)^3}.$$

The derivative is defined for all w and is only equal to zero when the numerator is zero. Factoring the common factor of $(w^2+225)^{1/2}$ gives

$$(w^2+225)^{1/2}(w^2+225-3w^2),$$

and since (w^2+225) is never zero, we must have

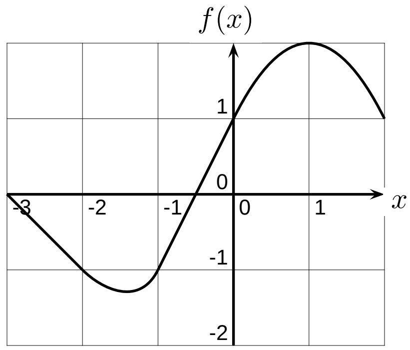
$$2w^2 = 225,$$

or

$$w = \pm\sqrt{\frac{225}{2}} = \pm\frac{15}{\sqrt{2}}.$$

Since w is a length, we discard the negative root, and now must test the one critical point $w = \frac{15}{\sqrt{2}}$. Note that for $w < \frac{15}{\sqrt{2}}$, the first derivative is positive, and for $w > \frac{15}{\sqrt{2}}$ the derivative is negative. Thus, by the first derivative test, $w = \frac{15}{\sqrt{2}}$ is a local maximum. Since the function is continuous and this is the only critical point on the domain, $w = \frac{15}{\sqrt{2}}$ is the global maximum. The mic should be placed $\frac{15}{\sqrt{2}} \approx 10.6$ feet from the wall to maximize the sensitivity to sounds at X.

6. (12 points) The graph of a function f is shown below, together with a table of values for its derivative f'. Let $g(x) = f(f(x))$.



x	$f'(x)$
-3	-1
-2	-1
-1	2
0	2
1	0
2	-2

(a) (2 points) Find $g(-2)$

$g(-2) = f(f(-2)) = f(-1) = -1$.

(b) (3 points) Find $g'(-2)$

By the chain rule, $g'(-2) = f'(f(-2)) \cdot f'(-2) = f'(-1) \cdot (-1) = (2)(-1) = -2$.

(c) (3 points) Write an expression for $g''(x)$ in terms of f and its derivatives.

Again by the chain rule, $g'(x) = f'(f(x)) \cdot f'(x)$, so

$$g''(x) = f'(f(x)) \cdot f''(x) + f''(f(x)) \cdot f'(x) \cdot f'(x).$$

(d) (4 points) Suppose $f''(-1) = 2$. What is $g''(-1)$?

Using the answer above,

$$\begin{aligned} g''(-1) &= f'(f(-1)) \cdot f''(-1) + f''(f(-1)) \cdot f'(-1) \cdot f'(-1) \\ &= (2) \cdot (2) + (2) \cdot (2)^2 \\ &= 4 + 8 = 12. \end{aligned}$$

7. (a) (4 points) Show that the point $(x, y) = (3, -6)$ lies on the curve defined by $y^2 - x^3 - x^2 = 0$.

Check: $(-6)^2 - 3^3 - 3^2 = 36 - 27 - 9 = 0$.

(b) (4 points) What is the equation of the tangent line to the curve at the point $(3, -6)$?

Use implicit differentiation to get a formula for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} - 3x^2 - 2x = 0,$$

so

$$\frac{dy}{dx} = \frac{3x^2 + 2x}{2y}.$$

Evaluating at $(x, y) = (3, -6)$, we get $\frac{dy}{dx} = \frac{27+6}{-12} = -\frac{33}{12}$. This is the slope of the tangent line, so its equation is

$$y = -\frac{33}{12}(x - 3) - 6.$$

(c) (2 points) Consider the function $f(x) = x\sqrt{x+1}$. What is the domain of f?

The function f is defined whenever $\sqrt{x+1}$ is defined, that is, when $x + 1 \geq 0$, or $x \geq -1$. [Note that f is one of the explicitly defined functions for the curve above by solving for y and restricting y to the positive root.]

(d) (6 points) Find all critical points, local maxima, and local minima of f. Which of the local maxima and minima are global maxima / minima?

First, take the derivative:

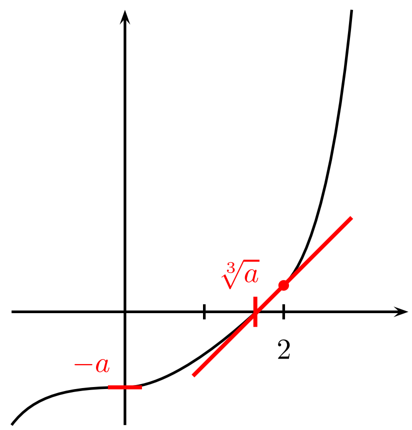
$$\begin{aligned} f'(x) &= \sqrt{x+1} + \frac{1}{2}\frac{x}{\sqrt{x+1}} \\ &= \frac{2(x+1) + x}{2\sqrt{x+1}} \\ &= \frac{3x+2}{2\sqrt{x+1}}. \end{aligned}$$

The derivative is 0 when $3x + 2 = 0$, *i.e.*, when $x = -\frac{2}{3}$. The derivative is undefined for $x = -1$, which is in the domain of f.

Thus the critical points are $x = -\frac{2}{3}$ and $x = -1$. To check if these are local maxima or minima, we can apply the first derivative test. Near $-\frac{2}{3}$, we have $3x + 2 < 0$ if $x < -\frac{2}{3}$ and $3x + 2 > 0$ if $x > -\frac{2}{3}$. Since $f'(x)$ has the same sign as $3x + 2$, we conclude that f has a local minimum at $x = -\frac{2}{3}$. As for the endpoint -1, for $x > -1$ but near -1, we have $3x + 2 < 0$, so $f'(x) < 0$. Therefore f has a local maximum at $x = -1$. Note that $f(x) \to \infty$ as $x \to \infty$, so the local minimum at $x = -\frac{2}{3}$ is a global minimum. There is no global maximum.

8. Let $f(x) = x^3 - a$, for $a > 1$ a constant. The graph of f is shown below.

   (a) (2 points) Find and label the numbers $-a$ and $\sqrt[3]{a}$ on the axes below.



   (b) (4 points) Find the equation for $L(x)$, the linear approximation to f near $x = 2$. Your equation will contain the constant a. Sketch the graph of $L(x)$ on the axes above.

   In general, the equation for L is $L(x) = f'(2)(x-2) + f(2)$. Since $f(2) = 2^3 - a = 8 - a$, and $f'(2) = 3(2)^2 = 12$, we get

$$L(x) = 12(x-2) + 8 - a = 12x - 16 - a.$$

   (c) (4 points) Use the function L to approximate 2.01^3.

   Since $f(x) = x^3 - a$, we see $x^3 = f(x) + a$. Using L to approximate f, we get $x^3 \approx L(x) + a = 12x - 16$ for x near 2. So

$$2.01^3 \approx 12(2.01) - 16 = 8.12.$$