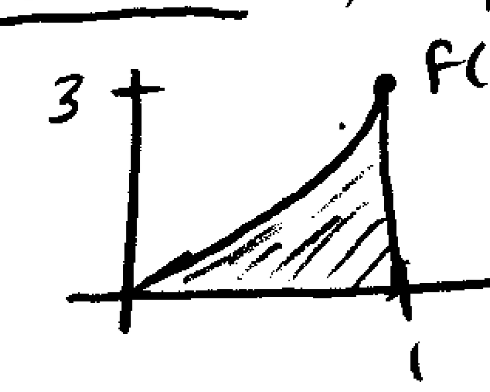
1. (2 points each) For each of the following, circle all the statements which are **always** true. For the cases below, one statement may be true, **or** both **or** neither of the statements may be true.

(a) Let $f(t) = t^2 + 2t$.

- $\frac{d}{dt}\int_0^1 f(t)\,dt = t^2 + 2t.$
- $\frac{d}{dt}\int_0^1 f(t)\,dt = 0.$

$\int_0^1 f(t)\,dt$ is a constant, equal to this area:



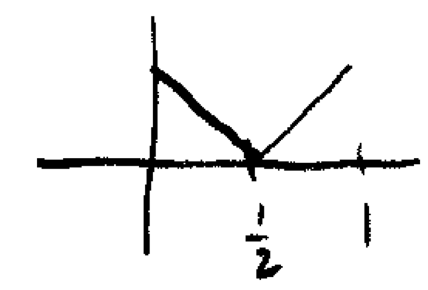
(b) Let $g(x)$ be continuous on the interval $[0, 1]$.

- The limit $\lim_{n\to\infty}\left(\sum_{k=1}^{n} g(x_k)\cdot\frac{1}{n}\right)$ exists.

this is $\int_0^1 g(x)\,dx$, which always exists

- The limit $\lim_{h\to 0}\frac{g(0.5+h)-g(0.5)}{h}$ exists.

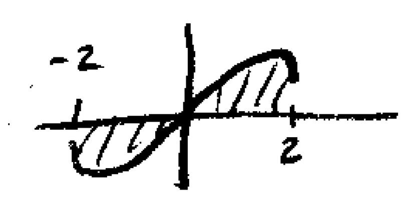
This is the derivative $g'(.5)$. It might exist, but might not, e.g. $g(x) = |x - \frac{1}{2}|$.



(c) Suppose $\int_{-2}^{2} F(x)\,dx = 5$.

- F is not an odd function.

if it were, the integral would be 0



- For some c in the interval $[-2, 2]$, $F(c) > 1$.

if $F(c)$ were ≤ 1 on the whole interval, then we would have $\int_{-2}^{2} F(t) \leq \int_{-2}^{2} 1\,dt = 4$.

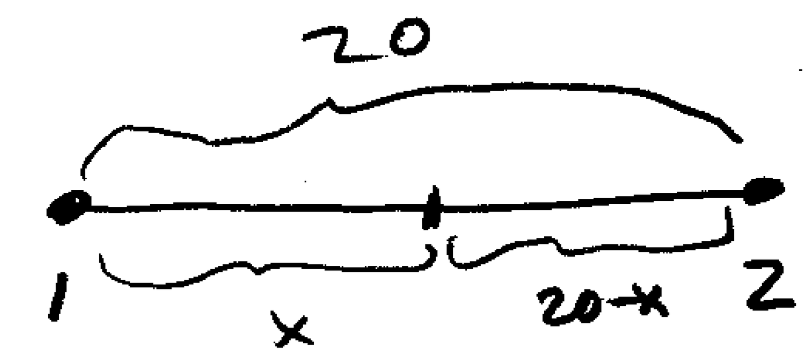
(d) Suppose h is a differentiable function defined on $[a, b]$, with antiderivative H. Assume $h(t) > 0$ for all t in $[a, b]$

- h has either a local maximum or minimum (or both) on (a, b).
- H has a local maximum at b.

$H' = h > 0$ means H is increasing, so its max is at b.

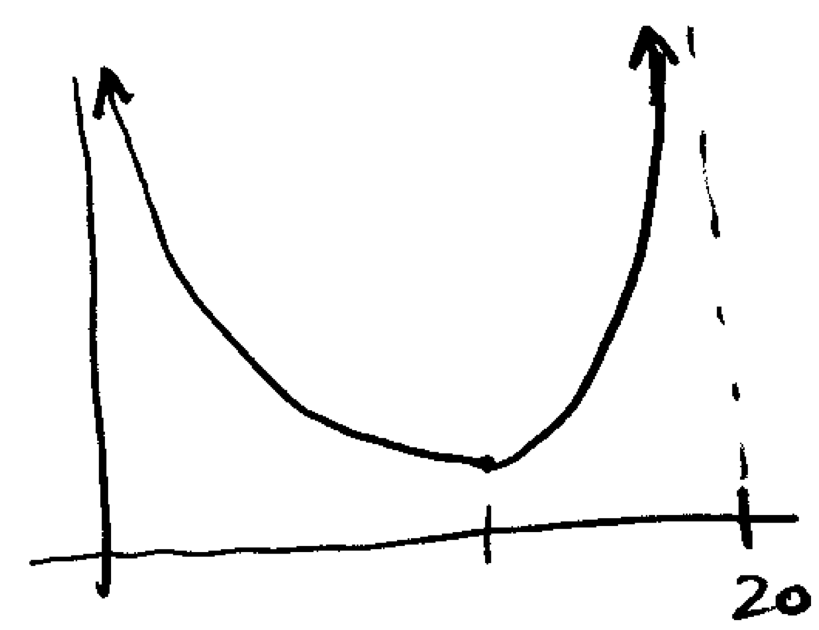
2. (10 points) A smokestack deposits soot on the ground with a concentration inversely proportional to the square of the distance from the stack. With two smokestacks 20 miles apart, the concentration of the combined deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{k_1}{x^2} + \frac{k_2}{(20-x)^2}$$



where k_1 and k_2 are positive constants which depend on the quantity of smoke each stack is emitting. If $k_1 = 7k_2$, find the point on the line joining the stacks where the concentration of the deposit is a minimum.

Graph From Calculator



$$S = k_1 x^{-2} + k_2(20-x)^{-2}$$

$$\frac{dS}{dx} = -2k_1 x^{-3} - 2k_2(20-x)^{-3}(-1)$$

$$= 2\left(k_2(20-x)^{-3} - k_1 x^{-3}\right).$$

So

$$\frac{dS}{dx} = 0 \Rightarrow k_2(20-x)^{-3} = k_1 x^{-3}$$

$$\Rightarrow \frac{k_2}{k_1} = \frac{x^{-3}}{(20-x)^{-3}} = \left[\frac{x}{20-x}\right]^{-3} = \left[\frac{20-x}{x}\right]^3$$

$$\Rightarrow \sqrt[3]{\frac{k_2}{k_1}} = \frac{20-x}{x} = \frac{20}{x} - 1$$

$$\Rightarrow 1 + \sqrt[3]{\frac{k_2}{k_1}} = \frac{20}{x} \Rightarrow x = \boxed{\frac{20}{1+\sqrt[3]{k_2/k_1}}}.$$

Check: 2nd derivative test

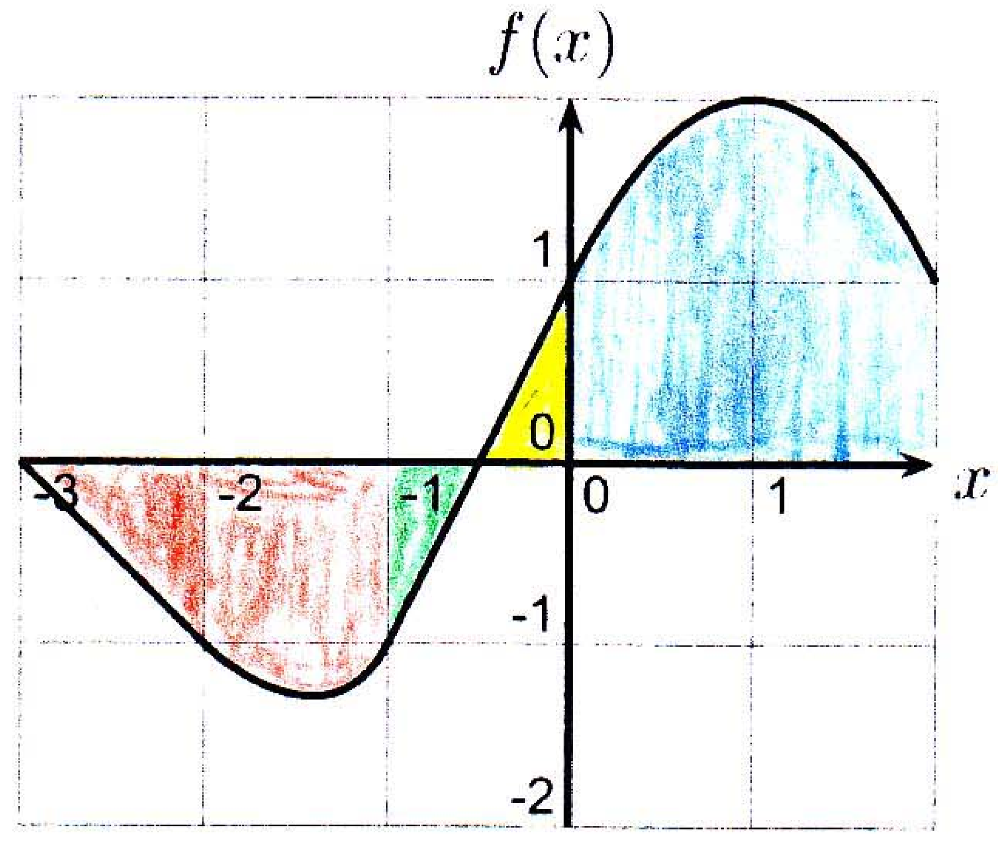
$$\frac{d^2S}{dx^2} = 2\left[-3k_2(20-x)^{-4}(-1) - (-3k_1x^{-4})\right]$$

$$= 6\left[\frac{k_2}{(20-x)^4} + \frac{k_1}{x^4}\right] > 0$$ Conc up $\Rightarrow$ <u>max</u>.

If $k_1 = 7k_2$, smokestack 1 is 7 times as bad as the other. And indeed, the min occurs at

$$x = \frac{20}{1+\sqrt[3]{1/7}} \approx \boxed{13.13},$$ closer to #2 than #1

3. (8 points) The graph of a function f is shown below.



Using the information given, write the following numbers in order from least to greatest:

$A = f'(1)$, $=$ slope @ $x=1$ ≈ 2

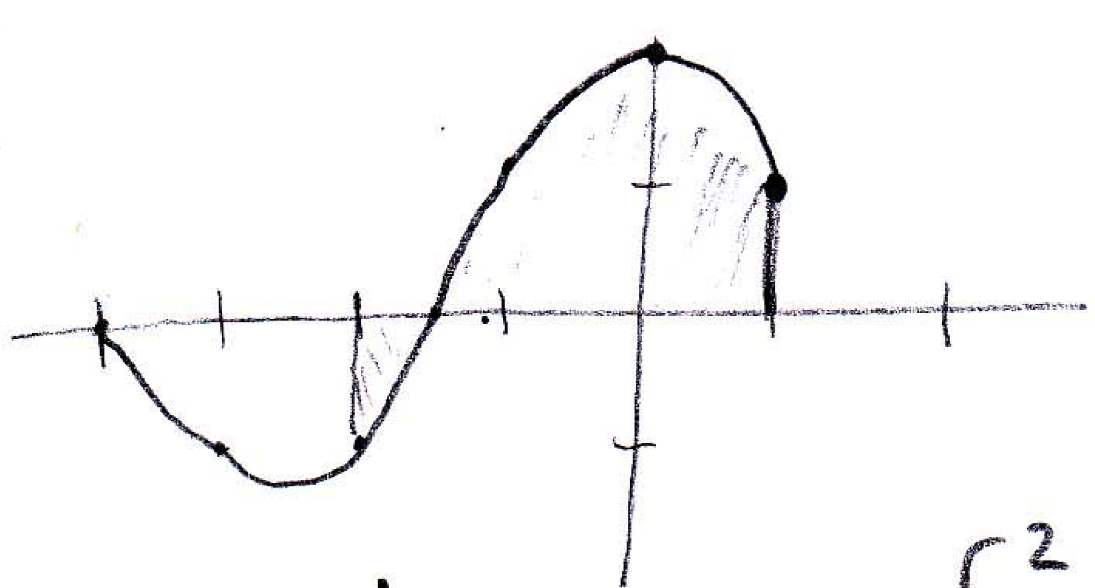
$B = \int_{-3}^{0} f(t)\,dt$, $=$ - Red - ~~Green~~ + ~~Yellow~~ ≈ -1.75

$C = \int_{-2}^{1} f(t+1)\,dt$, $=$ - ~~Green~~ + ~~Yellow~~ + Blue ≈ 3.5

$D = \int_{-1}^{2} (f(t)+1)\,dt.$ $= \int_{-1}^{2} f(t)\,dt + \int_{-1}^{2} 1\,dt$

$\approx \underbrace{3.5}_{\text{from part C}} + 3 = 6.5$

$f(t+1)$ is f shifted left 1:



So $\int_{-2}^{1} f(t+1)\,dt = \int_{-1}^{2} f(t)\,dt$

shift limits by 1

B < A < C < D

in general:

$$\int_a^b f(t+c)\,dt = \int_{a+c}^{b+c} f(t)\,dt$$

4. After seeing a Tigers game in Detroit, you get onto I-94 toward Ann Arbor at 10:00 pm. On I-94, you run into a few pockets of traffic. Glancing at the speedometer occasionally, you note your speed fluctuates as in the table below.

minutes after 10:00	0	5	10	15	20	25	30
speed (mph)	50	71	65	42	35	70	74

(a) (4 points) Using the information given, estimate your acceleration at 10:10. Be sure to include units.

$$\text{accel}(10) \approx \frac{\text{Speed}(10:15) - \text{speed}(10:05)}{10:15 - 10:05} = \frac{42-71 \text{ mph}}{10 \text{ min}}.$$

$$= \boxed{-2.9 \text{ mph}/\text{min}}$$

(b) (6 points) Approximately how far from your entrance onto I-94 are you at 10:30? Be certain to show how you arrived at your approximation, and again, give correct units.

$$LHS = 5\text{min}\left[50 + 71 + 65 + 42 + 35 + 70\right]\text{mph} \cdot \frac{1\text{hr}}{60\text{min}}$$

$$= \frac{5}{60}\left[313\right] \text{miles} = 27.75$$

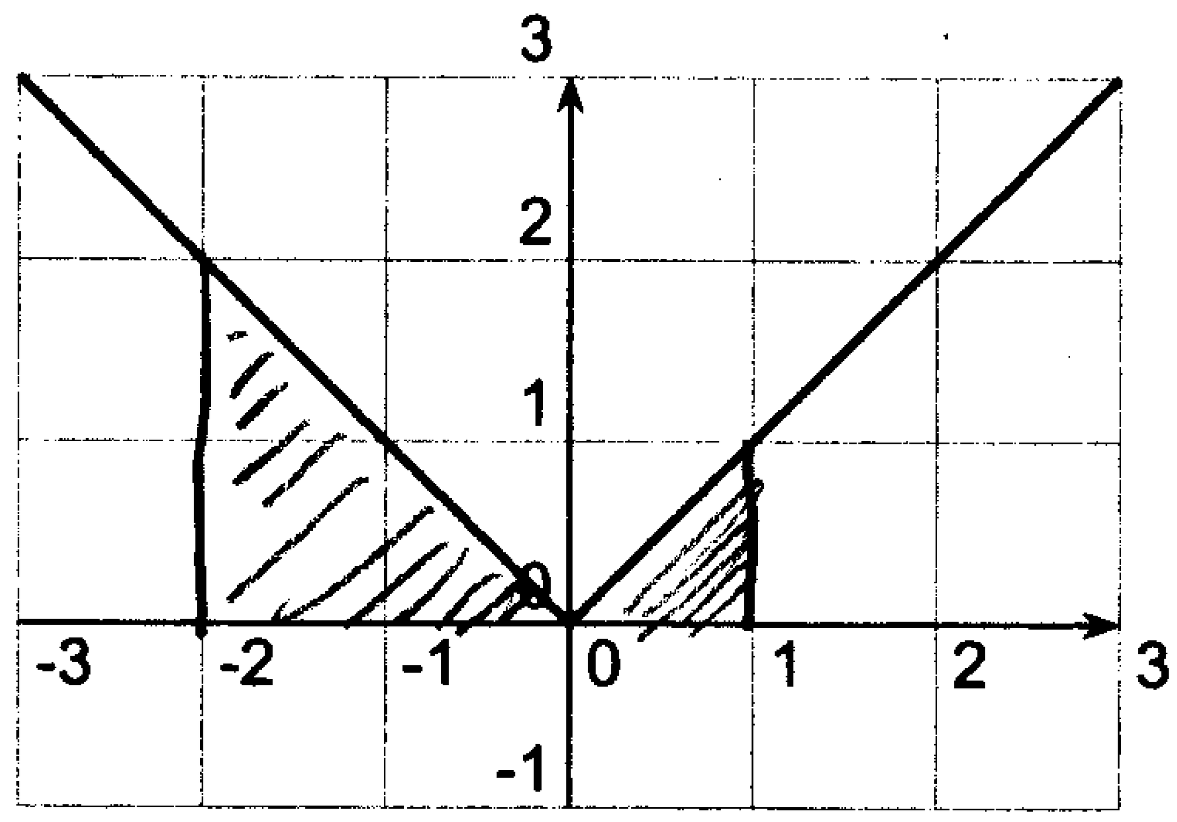
$$RHS = \frac{5}{60}\left[71+65+42+35+70+74\right]\text{miles}$$

$$= 29.75 \text{ miles}$$

So you have gone approximately $\boxed{28.75 \text{ miles}}$

5. Let $f(x) = |x|$.

(a) (4 points) Find $\int_{-2}^{1} f(x)\,dx$ using geometry (i.e., areas). Show your work on the graph below and circle your numerical answer.



$$\frac{1}{2}(2)(2) + \frac{1}{2}(1)(1) = \frac{1}{2}(4+1) = \boxed{\frac{5}{2}}$$

(b) (4 points) Find a formula for an antiderivative of $f(x)$, given the piecewise formula

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

$$\boxed{F(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 & \text{if } x < 0 \end{cases}}$$

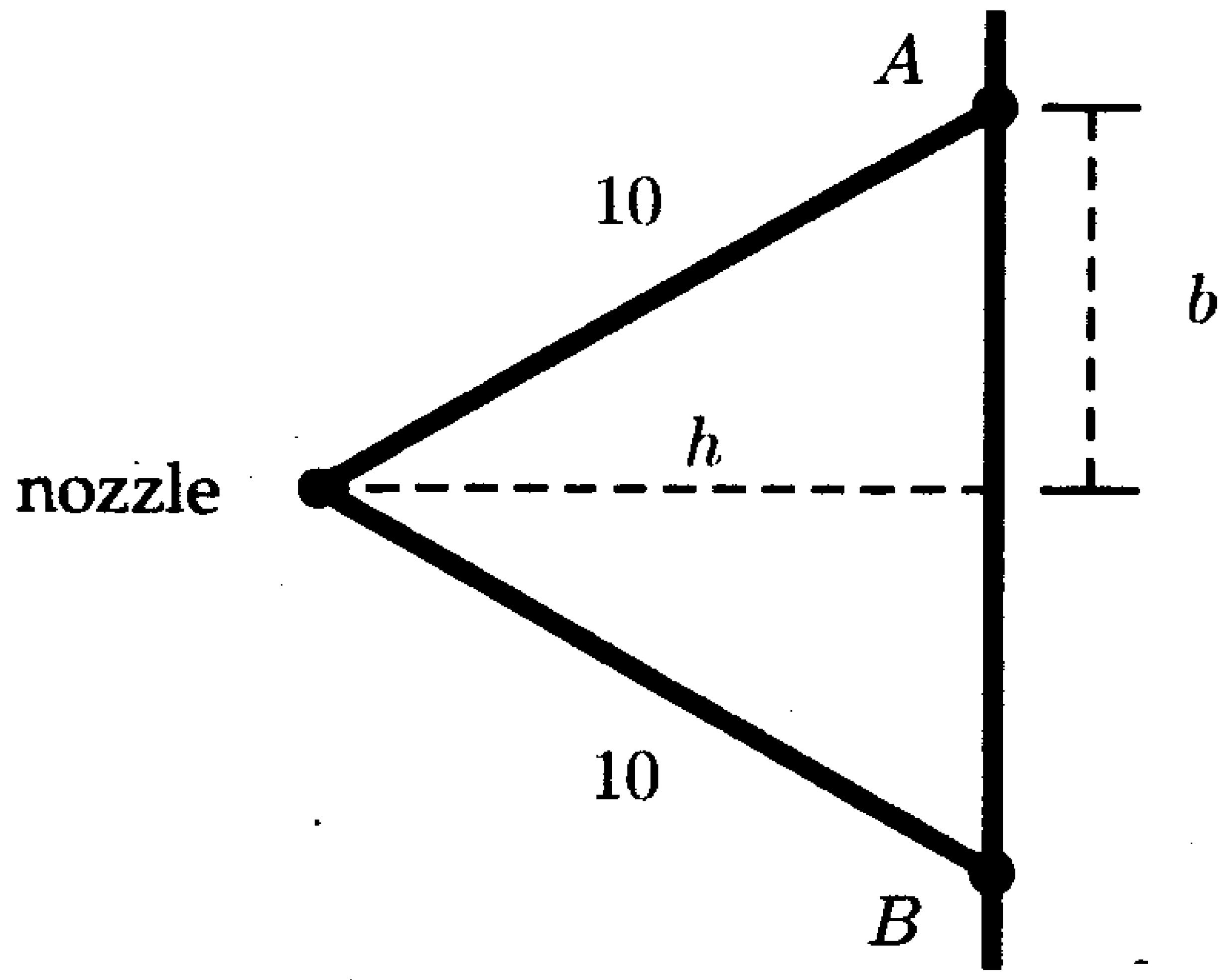
(c) (4 points) Using the Fundamental Theorem and your answer to (5b), compute $\int_{-2}^{1} f(x)\,dx$.

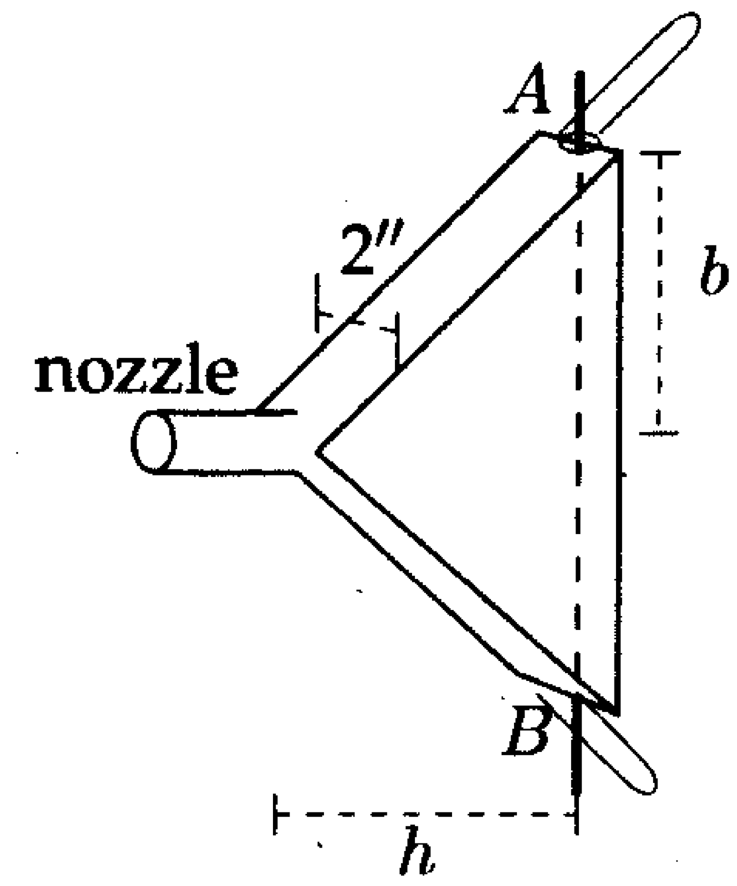
$$\int_{-2}^{1} f(x)\,dx = F(1) - F(-2) = \frac{1}{2}(1)^2 - \left(-\frac{1}{2}(-2)^2\right)$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 4 = \boxed{\frac{5}{2}}$$

W08 RR

6. (10 points) A bellows[1] has a triangular frame made of three rigid pieces. Two pieces, each 10 inches long, are hinged at the nozzle. They are attached to the third piece at points A and B which can slide, as shown in the diagrams below (the figure to the right shows a 3D sketch of the bellows; the figure to the left, a 2D sketch that may be specifically useful for solving the problem).





Each piece of the frame is 2 inches wide, so the volume (in cubic inches) of air inside the bellows is equal to the area (in square inches) of the triangular cross-section shown above times the height of 2. Suppose you pump the bellows by moving A downward towards the center at a constant speed of 3 in/s. (So B also moves upwards at the same speed.) What is the rate at which air is being pumped out when A and B are 12 inches apart? (So A is 6 inches from the center of the vertical piece of the frame.)

We have:

$$b^2+h^2=10^2$$

$$V=\text{Vol}=2\cdot\tfrac{1}{2}(2b)h=2bh$$

We want:

$$\frac{dV}{dt} \text{ when } \frac{db}{dt}=-3\text{ in/sec}, \quad b=6\text{ in}$$

Eliminate h:

$$h=\sqrt{100-b^2}$$

So

$$V=2b\sqrt{100-b^2}$$

$$\Rightarrow \frac{dV}{dt}=2\left[b\cdot\tfrac{1}{2}(100-b^2)^{-1/2}(-2b)+(1)\cdot(100-b^2)^{1/2}\right]\cdot\frac{db}{dt}$$

Now plug in:

$$\frac{dV}{dt}=2\left[6\cdot\tfrac{1}{2}(100-6^2)^{-1/2}(-2\cdot6)+(100-6^2)^{1/2}\right](-3)$$

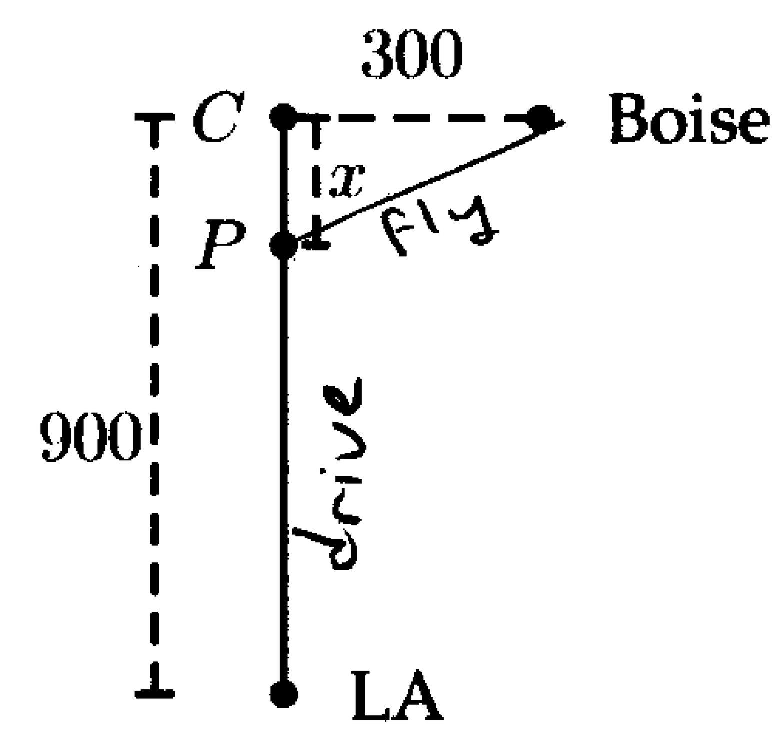
$$=-6\left[-36\cdot64^{-1/2}+64^{1/2}\right]$$

$$=-6\left[\frac{-36}{8}+8\right]=\boxed{-21\text{ in}^3/\text{sec}}.$$

[1] A bellows is a device with a nozzle attached to a chamber; it is used to blow air out through the nozzle by reducing the volume of the chamber. In the bellows described here this is accomplished by moving the points A and B as indicated.

W08 OPT

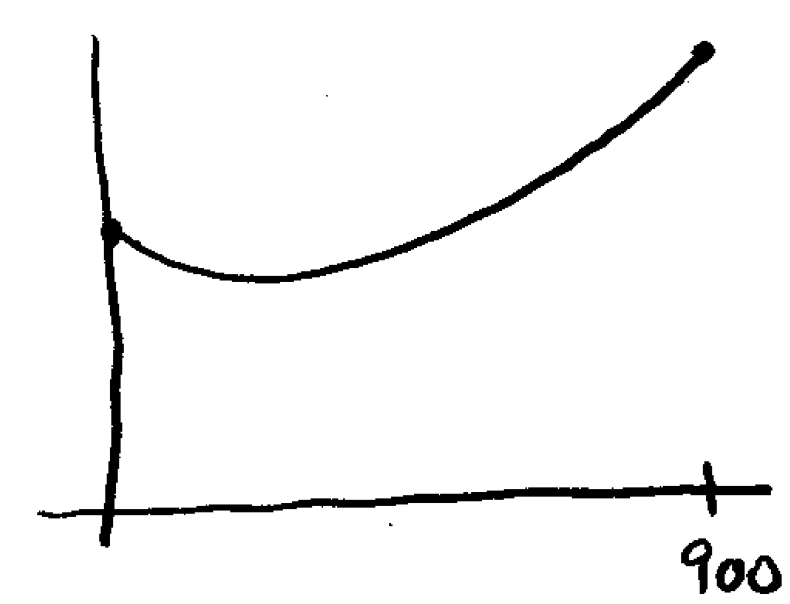
7. (12 points) You decide to take your two-week vacation from your job at the Idaho Potato Company in Boise to drive on California's coastal highway and visit friends in Los Angeles.. Your plan is to fly a small plane to the coast, land at point P, and drive a rented car from there to LA. Assume the coast highway is straight, that C is the point on the coast directly west of Boise, and the distances (in miles) are as in the diagram below. (That is, ignore whatever actual experience you may have of that road!)



You want to minimize gas costs: it costs 30 cents per mile to fly, and 10 cents per mile to drive. Set up the function (of x) which describes the cost of gas for the trip, and find the position P where you should land to minimize the cost.

$$C = \text{total cost} = .30\sqrt{300^2+x^2} + .10(900-x)$$

Graph from Calc:



$$\frac{dC}{dx} = .30 \cdot \frac{1}{2}(300^2+x^2)^{-1/2}(2x) + .10(-1)$$

$$= \frac{.30x}{\sqrt{300^2+x^2}} - .10$$

So $\frac{dC}{dx} = 0 \Rightarrow \frac{.30x}{\sqrt{300^2+x^2}} = .10 \Rightarrow 3x = \sqrt{300^2+x^2}$

$$\Rightarrow 9x^2 = 300^2 + x^2 \Rightarrow 8x^2 = 300^2$$

$\Rightarrow x = \pm\frac{300}{\sqrt{8}}$. It doesn't make sense for x to be negative (why fly north?), so

$$x = \frac{300}{\sqrt{8}} = 75\sqrt{2} \approx \boxed{106.7 \text{ miles}}.$$

Check: Cost @ crit pt is \$174.85

Min must be at crit pt or at endpoints.

Endpoints: $x = 0 \Rightarrow C = .30\sqrt{300^2+0^2} + .10(900-0) = \$180$

$x = 900 \Rightarrow C = .30\sqrt{300^2+900^2} + 0 = \$284.60$

so crit pt is minimum.

W08 INT

8. (3 points each) Let

- $\int_a^b f(x)dx = 8$, and $\int_a^b (f(x))^2 dx = 12$,
- $\int_a^b g(t)dt = 2$, and $\int_a^b (g(t))^2 dt = 3$.

Evaluate the following integrals, if the value can be determined. If there is information missing, clearly state what is missing.

(a) $\int_a^b (f(x) + g(x))dx$ $= \int_a^b f(x)dx + \int_a^b g(x)dx = 8 + 2 = \boxed{10}$

(b) $\int_a^b cf(z)dz$, for c a constant $= c\int_a^b f(z)dz = \boxed{8c}$

(c) $\int_a^b ((f(x))^2 - g(x^2))dx$ $= \int_a^b (f(x))^2 dx - \int_a^b g(x^2)dx$

$= 12 - \underbrace{\int_a^b g(x^2)dx}_{\text{But we don't know this!}}$

9. (6 points) Find $\int_2^5 f(x)dx$, if $\int_2^5 (3f(x) + 4)dx = 18$.

Let $I = \int_2^5 f(x)dx$

$18 = \int_2^5 (3f(x) + 4)dx = \int_2^5 3f(x)dx + \int_2^5 4dx$

$= 3\int_2^5 f(x)dx + 4(5-2) = 3I + 12.$

So $I = \dfrac{18-12}{3} = \boxed{2}.$

10. An osprey is flying 400 feet above the sea when it drops a fish it just caught. The fish falls with constant downward acceleration 32 ft/s^2 toward the water.

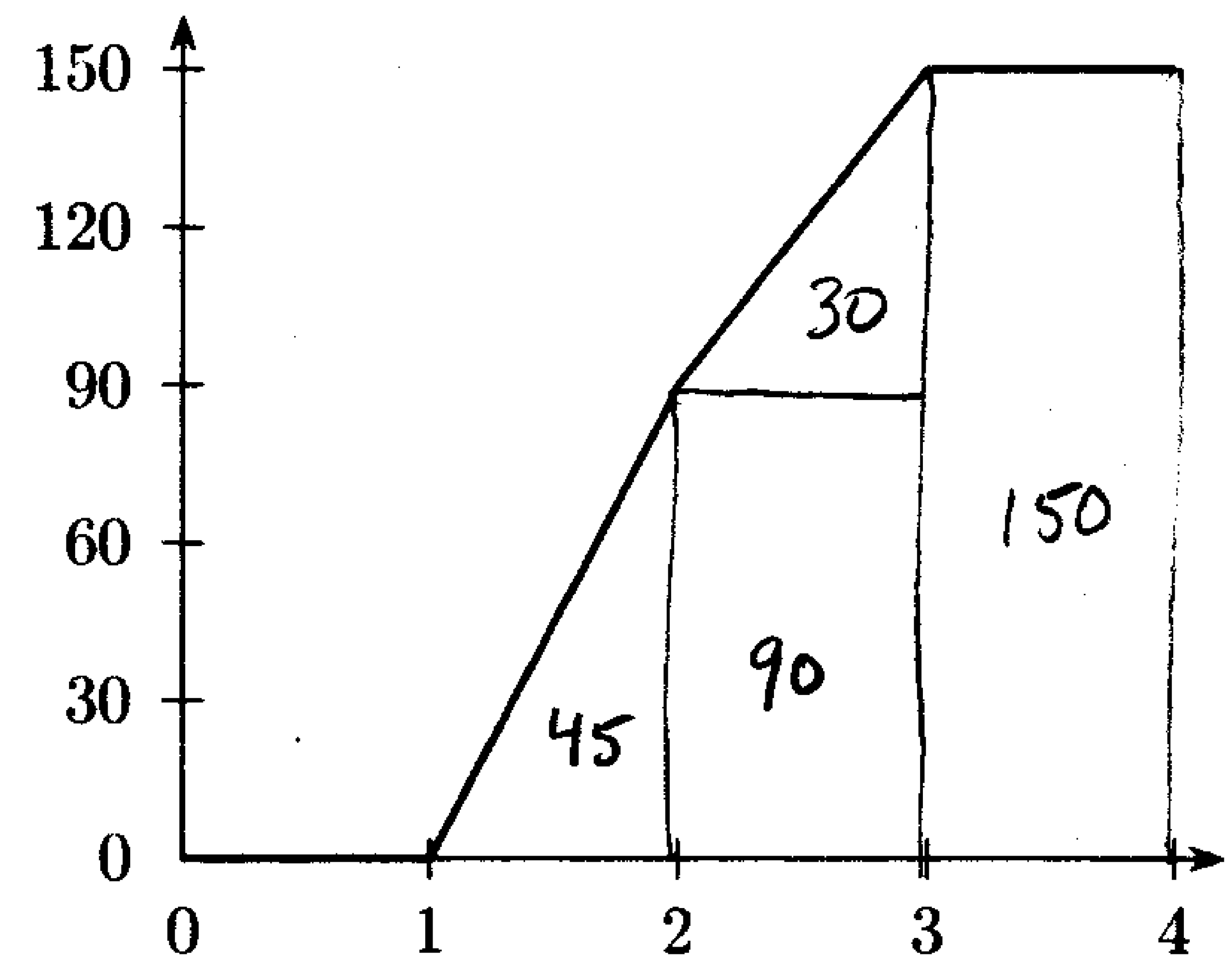
(a) (3 points) Write an expression for the function $F(t)$ giving the distance (in feet) the fish has fallen t seconds after being dropped.

accel = 32

velocity = $\int$ accel = $32t$

distance = $\int$ velocity = $32 \cdot \left(\frac{1}{2}t^2\right) = \boxed{16t^2}$

The osprey begins to dive 1 second after dropping the fish, and its downward velocity is given, in ft/s, by the function $V(t)$, where t is the number of seconds after the osprey dropped the fish. The graph of $V(t)$ is shown below.



(b) (9 points) Write an integral expressing the osprey's average velocity *during its dive*, and evaluate it using the graph.

avg velocity = $\frac{1}{4-1}\int_{1}^{4} V(t)\,dt = \frac{1}{3}\left[45+90+30+150\right] = \boxed{105 \text{ ft/sec}}$

dive starts at $t=1$

(c) (3 points) Based on the information given, it is possible for the osprey to recover the fish before it hits the water? Explain.

at $t=4$, the fish has fallen $16(4^2) = 256$ ft, and the bird has fallen $45+90+30+150 = 315$ ft.

So $\boxed{\text{yes}}$, at some point the bird passed the fish.