MATH 115 –FIRST MIDTERM

February 10, 2009

NAME: ______________SOLUTIONS______________

INSTRUCTOR: ______________ SECTION NUMBER: ______________

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1. **Do not open this exam until you are told to begin.**
2. This exam has 10 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	9	
2	16	
3	13	
4	10	
5	14	
6	11	
7	7	
8	8	
9	12	
TOTAL	100	

1. Air pressure, P, decreases exponentially with the height, h, in meters above sea level. The unit of air pressure is called an *atmosphere*; at sea level, the air pressure is 1 atm.

   (a) (5 points) On top of Mount McKinley, at a height of 6198 meters above sea level, the air pressure is approximately 0.48 atm. Use this to determine the air pressure 12 km above sea level, the maximum cruising altitude of a commercial jet.

   Since we know P is a decreasing exponential function of h,

   $$P(h) = P_0\, e^{kh}$$

   for some constants k and P_0. Plugging in $h = 0$, we find that $P_0 = 1$. Plugging in $h = 6198$, we obtain the equation

   $$0.48 = e^{6198k}.$$

   Taking the natural logarithm of both sides and dividing by 6198, we find $k \approx -1.184 \cdot 10^{-4}$. Using this value of k, we find that

   $$P(12000) = e^{-12000\,k} \approx 0.241 \text{ atm}.$$

   (b) (4 points) Determine $P^{-1}(0.7)$. Include units!

   We know that $P^{-1}(0.7)$ gives us the height above sea level at which the air pressure is 0.7 atm. Thus, we want to solve the equation

   $$0.7 = e^{-1.184 \cdot 10^{-4} h}$$

   for h. Thus,

   $$\ln 0.7 = -1.184 \cdot 10^{-4} h,$$

   so $h = \frac{\ln 0.7}{-1.184 \cdot 10^{-4}} \approx 3012.46$ m (or ≈ 3.012 km).

2. A new company produces and sells socks. By far their most successful item is the business sock (that's why they call them business socks), and the company hires a young consultant to assess the impact of advertising this popular product. Let S denote the yearly sales revenue, in thousands of dollars, and a denote the annual advertising expenditure, also in thousands of dollars. The company assumes that sales revenue will depend on advertising, so we write $S = f(a)$.

   (a) (2 points) What does the company hope is true about the sign of f'? Explain.

   It would be reasonable for the company to expect $f' > 0$, since this simply means that they sell more socks as they spend more on advertising.

   (b) (2 points) The consultant suggests that $\lim_{a\to\infty} f'(a) = 0$. Is this reasonable? Why or why not?

   This is a reasonable assumption, since eventually all the people potentially interested in purchasing business socks will have seen an advertisement; putting more money in at that point doesn't change the situation much. In the language of economics, marginal revenue is decreasing. [Note: other answers are accepted–with reasonable justification.]

   (c) The consultant makes the following statements. Interpret her observations in practical terms. Do not use the word "rate"!

   i. (3 points) $f(0) = 3$

   Even if the company doesn't spend any money on advertising, they will sell \$3000 worth of business socks per year.

   ii. (3 points) $f'(0) = 4$

   If the company spends \$1000 on advertising, their sales revenue should increase by approximately \$4000.

   iii. (3 points) $f^{-1}(6.6) = 1$

   If the company spends \$1000 on advertising, the sales revenue will be \$6600.

   iv. (3 points) $\left(f^{-1}\right)'(6.6) = 0.31$

   When the company's annual sales revenue is \$6600, increasing the sales revenue to \$7600 will require increasing spending on advertising by approximately \$310.

3. Table 1 below displays some values of an invertible function $f(x)$, while Figure 2 depicts the graph of the function $g(x)$.

Table 1

x	0	1	2	3	4	5	6
$f(x)$	2	6	5	4	1	3	7

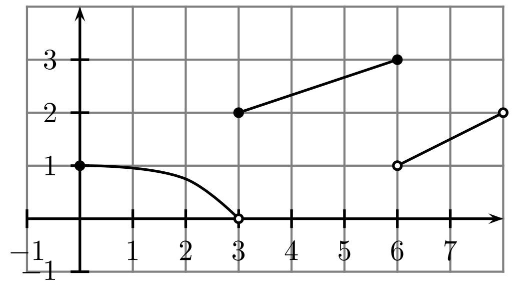


Figure 2: Graph of $g(x)$

(a) (4 points) What is the domain of g? of g^{-1}?

The domain of g is $[0, 8)$.
The domain of g' is $(0, 3]$.

(b) (1 point each) Evaluate the following:

i. $g(5)$

¿From the graph, $g(x)$ is linear on $[3, 6]$, with slope $\frac{1}{3}$. Thus,

$$g(5) = g(3) + 2 \cdot \frac{1}{3} = 2 + \frac{2}{3} = \frac{8}{3}$$

ii. $g\big(g(6)\big)$

¿From the graph, $g(6) = 3$, so

$$g\big(g(6)\big) = g(3) = 2$$

iii. $\lim_{x \to 3^-} g(x)$

$\lim_{x \to 3^-} g(x) = 0$

iv. $g^{-1}\big(f^{-1}(5)\big)$

¿From the table $f^{-1}(5) = 2$, so

$$g^{-1}\big(f^{-1}(5)\big) = g^{-1}(2) = 3$$

v. $f\big(f(5)\big)$

¿From the table, $f(5) = 3$, so

$$f\big(f(5)\big) = f(3) = 4$$

(c) (4 points) Order the following from smallest to largest: $g'(1), g'(2), g'(5), g'(6.4)$.

$$g'\left(\underline{2}\right) < g'\left(\underline{1}\right) < g'\left(\underline{5}\right) < g'\left(\underline{6.4}\right)$$

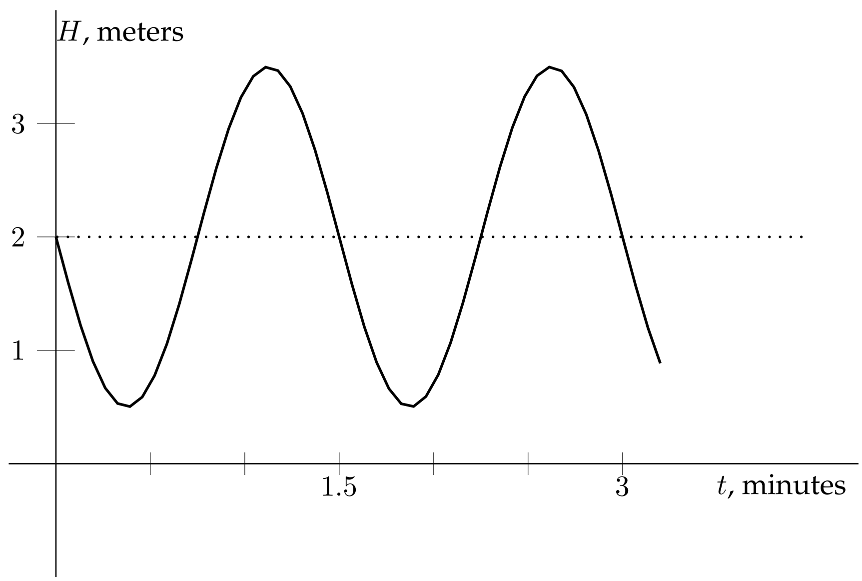
4. The Awkward Turtle is riding a mini ferris wheel! The wheel has radius 1.5 meters but is lifted off the ground, so that even when he is at the lowest point of the ride, the Awkward Turtle is still 0.5 meters above the ground, which is, needless to say, distinctly awkward. The wheel turns at a constant rate of 1 revolution every 90 seconds.

   Suppose that precisely at noon, the Awkward Turtle is 2 meters above the ground and moving toward the ground. Let $H(t)$ denote the height (in meters) of the Awkward Turtle above the ground, t minutes after noon.

   (a) (2 points) What is the sign of $H'(0)$? Explain.

   $H'(0)$ is negative, because the turtle is moving downwards at noon, i.e. the height is decreasing at noon.

   (b) (4 points) Sketch, on the axes below, the function $H(t)$. Make sure you label the tick-marks!



   (c) (4 points) Determine a formula for the function $H(t)$.

$$H(t) = -1.5 \sin\left(\frac{4\pi}{3} t\right) + 2$$

5. The graph on the left below (Figure 1) depicts a derivative function, f'. The graph indicates the full behavior of f' — *i.e.*, f' does not have changes in direction that are not shown in the figure.

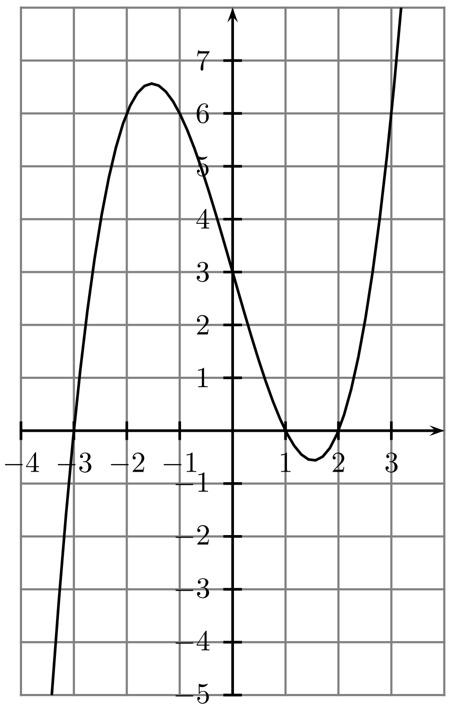


Figure 1: graph of f'

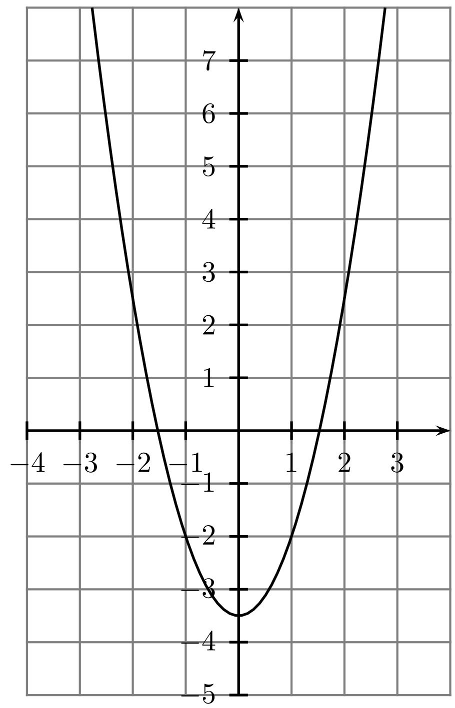


Figure 2: graph of f''

(a) (4 points) Using the axes provided in Figure 2 above, sketch a graph of $f''(x)$.

(b) (4 points) On which interval(s) is the original function f increasing?

On $[-3, 1]$ and $[2, \infty)$ (or with open intervals).

(c) (2 points) On which which interval(s) is f concave up?

On $(-\infty, -1.5]$ and $[1.5, \infty)$ (or with open intervals).

(d) (4 points) If $f(-2) = 3$, approximate $f(-1)$.

Since the slope at $f'(-2) = 6$, we have

$$f(-1) \approx f(-2) + 6 = 3 + 6 = 9$$

6. Let $f(x) = \sin x$ where x is in ***degrees***.

   (a) (4 points) Write down a formula for $f'(180)$ using the *limit* definition of the derivative.

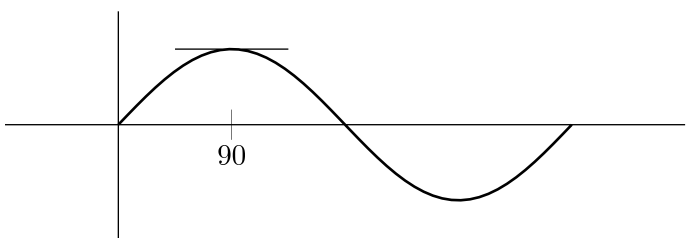
$$f'(180) = \lim_{h \to 0} \frac{\sin(180+h) - \sin(180)}{h} = \lim_{h \to 0} \frac{\sin(180+h)}{h}$$

   (b) (3 points) Use the *limit* definition to approximate $f'(180)$ to 3 decimals. Show how you obtained your answer.

Plugging $h = 0.1$ into $\frac{\sin(180+h)}{h}$, we find it is approximately -0.01745.
Plugging $h = 0.01$ yields the same approximation (-0.01745).
Also, letting $h = -0.01$, we have ≈ -0.01745.
Thus, we can say

$$f'(180) \approx -0.0175$$

   (c) (2 points) What is the exact value of $f'(90)$? Justify your answer geometrically.



The tangent line at $x = 90$ is horizontal, so $f'(90) = 0$.

   (d) (2 points) Let $g(x) = \sin x$ where x is in radians. Determine a continuous function $h(x)$ such that for all x, $f(x) = g\big(h(x)\big)$.

$$h(x) = \frac{\pi}{180}\,x$$

[Note: This answer is not unique.]

7. Table 1 below shows some values of the function $f(x)$. Assume that both f' and f'' are defined on $[-1, 7]$.

Table 1

x	0	1	2	3	4	5	6
$f(x)$	-2	1	5	12	15	16	13

Table 2

x	0	1	2	3	4	5	6
$f'(x)$							

(a) (4 points) Use the data given in Table 1 to fill in approximate values of f' in Table 2. Possible answers (depending on whether one takes left, right, or averages) are:

x	0	1	2	3	4	5	6
$f'(x)$	3	4	7	3	1	-3	

x	0	1	2	3	4	5	6
$f'(x)$		3	4	7	3	1	-3

x	0	1	2	3	4	5	6
$f'(x)$		3.5	5.5	5	2	-1	

[Only the intermediate points on the table were checked on this portion of the problem.]

(b) (1 point) Where does the rate of change of f seem greatest?

Anywhere in $[2, 3]$ is acceptable.

(c) (2 points) What is the largest interval over which the table indicates that f is concave up?

$(0, 3)$

8. A continuous function f, defined for all x, is always decreasing and concave up. Suppose $f(6) = -6$ and $f'(6) = -1.5$.

(a) (2 points) How many zeros does f have? Justify your answer.

The function f is always decreasing, so it has at *most* one zero.

On the other hand, since it is concave up, f always lies above the line $y = -1.5\,x + 3$ (the line tangent to f at the point $(6, -6)$). In particular, f takes positive values. It also takes negative values (e.g. at $x = 6$) and is continuous, and so must cross the x-axis somewhere. Thus, f has at *least* one zero, and combining this with the first statement we conclude that f has precisely one zero.

(b) (2 points) Can $f'(2) = -1$? Justify your answer.

Since f is concave up, f' must be increasing. Since $f'(6) = -1.5$, we must have $f'(2) \leq -1.5$.

Therefore, the answer is NO.

(c) (4 points) Circle all intervals below in which f has at least one zero. Justify your choices with a picture and a short description.

i. $(-\infty, -6)$

ii. $[-6, -2)$

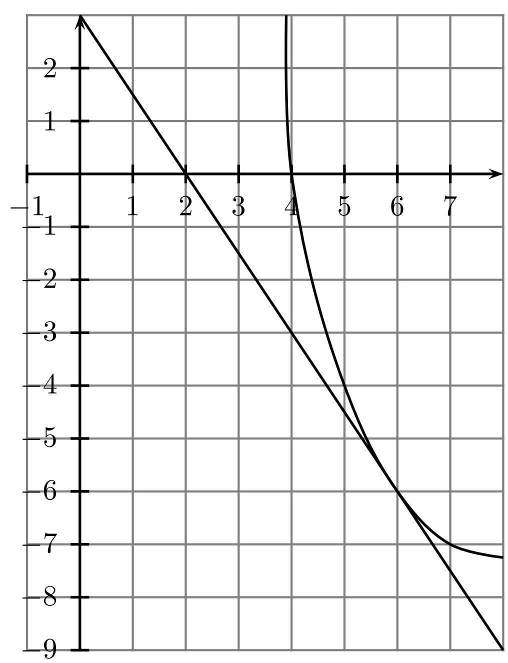
iii. $[-2, -1)$

iv. $[-1, 1)$

v. $[1, 2)$

vi. $[2, 6)$

vii. $[6, \infty)$



From the reasoning given for part (a), we know f lies above the line tangent to f at $x = 6$. Thus, the smallest x at which f could have a zero is at 2. If f decreases quickly enough, it could have a zero arbitrarily close to (but to the left of) $x = 6$.

9. (12 points) On the axes below, sketch a function f satisfying all the following properties. Be careful to label all important points on the axes.

- f has a vertical asymptote at $x = 4$
- f is continuous on $(-\infty, 4)$ and on $(4, \infty)$
- $f'(x) > 0$ and $f''(x) > 0$ for all x in $(-\infty, 0)$
- $f(0) = 2$
- f is not differentiable at $x = 0$
- $f''(x) > 0$ for all x in $(0, 2)$
- $f'(2) > 0$
- $f''(x) < 0$ for all x in $(2, 4)$
- For all $x > 4$, f is decreasing and is concave up
- $\lim_{x\to\infty} f(x) = 1$

One possible solution is shown below.