MATH 115 –SECOND MIDTERM

March 31, 2009

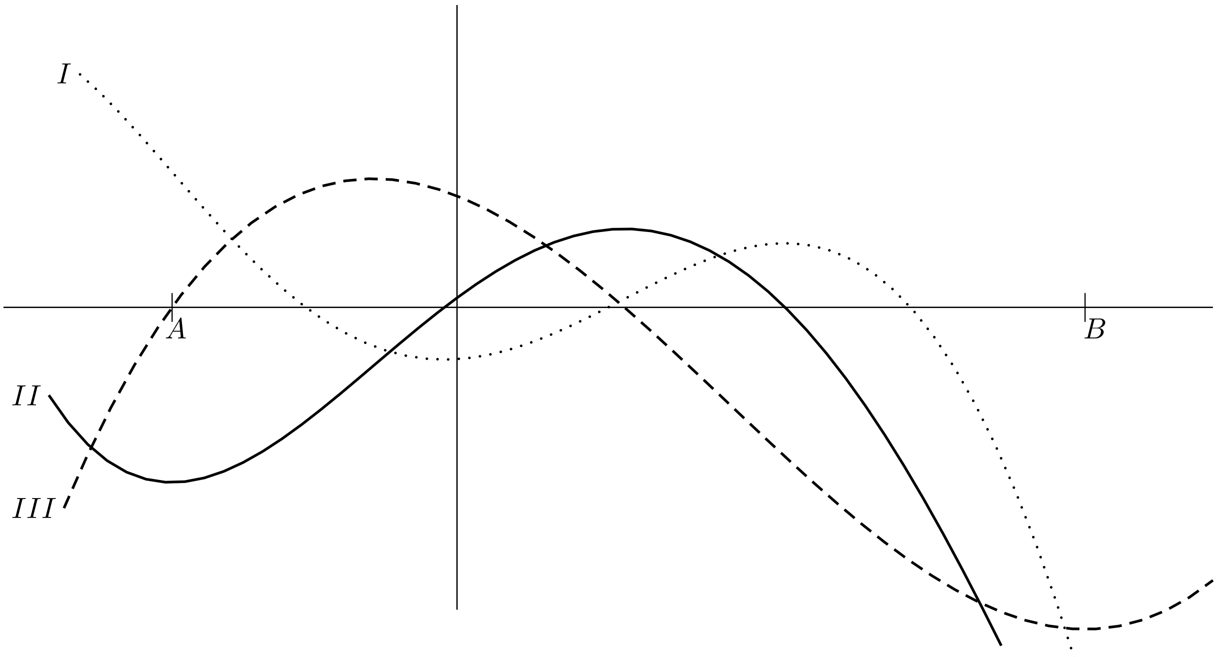
NAME: SOLUTIONS

INSTRUCTOR: ______________________ SECTION NUMBER: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	8	
2	12	
3	16	
4	16	
5	6	
6	6	
7	10	
8	10	
9	16	
TOTAL	100	

1. (8 points) On the axes below are graphed f, f', and f''. Determine which is which, and justify your response with a brief explanation.



I : f

II : f'

III : f''

Explanation:

The function III achieves a minimum at $x = B$ (marked in the figure above); since neither I nor II is 0 there, III must be f''. This tells us that f'' is 0 at the point marked A in the graph. Since the curve II achieves a minimum at $x = A$ while I clearly has non-zero derivative there, II must be f'. This in turn implies that I must be f.
[Note: there are many answers for this part of the question.]

2. (12 points) Suppose a is a positive (non-zero) constant, and consider the function

$$f(x) = \frac{1}{3}x^3 - 4a^2x.$$

Determine all maxima and minima of f in the interval $[-3a, 5a]$. For each, specify whether it is global or local.

We need to check values of f at the endpoints ($x = -3a$ and $x = 5a$) and wherever $f'(x) = 0$ or is undefined. Since $f'(x) = x^2 - 4a^2$, $f'(x)$ is defined for all x and $f'(x) = 0$ at $x = \pm 2a$. So, we have critical points $x = -2a, 2a$. We check all points individually to determine which are minima and which are maxima. We can use the second derivative, $f''(x) = 2x$ to help with the check.

- $\underline{x = -3a}$:
  $f'(-3a) = (-3a)^2 - 4a^2 = 5a^2 > 0$, so the function is increasing there, and this endpoint must be a (local) minimum. Since $f(-3a) = 3a^3$, we have the point $(-3a, 3a^3)$.
- $\underline{x = 5a}$:
  $f'(5a) = (5a)^2 - 4a^2 = 21a^2 > 0$, so this endpoint must be a (local) maximum. Since $f(5a) = \dfrac{65}{3}a^3$, we have the point $(5a, \dfrac{65}{3}a^3)$.
- $\underline{x = -2a}$:
  $f''(-2a) = -4a < 0$, so this must be a (local) maximum. Since $f(-2a) = \dfrac{16}{3}a^3$, we see that this occurs at the point $(-2a, \dfrac{16}{3}a^3)$.
- $\underline{x = 2a}$:
  $f''(2a) = 4a > 0$, so this must be a (local) minimum. Since $f(2a) = -\dfrac{16}{3}a^3$, we see that this minimum occurs at the point $(2a, -\dfrac{16}{3}a^3)$.

Comparing the y-values of the minima at $x = -3a, 2a$, we find that the global minimum occurs at $x = 2a$. Similarly, comparing the y-values of the maxima at $x = -2a, 5a$, we find the global maximum at the endpoint $x = 5a$.
Summing up:

- There's a local minimum at $(-3a, 3a^3)$.
- There's a local maximum at $(-2a, \dfrac{16}{3}a^3)$.
- There's a local and global minimum at $(2a, -\dfrac{16}{3}a^3)$.
- There's a local and global maximum at $(5a, \dfrac{65}{3}a^3)$.

3. Table 1 below displays some values of an invertible, differentiable function $f(x)$, while Figure 2 depicts the graph of the function $g(x)$. Set $h(x) = f\big(g(x)\big)$ and $j(x) = \dfrac{f(x)}{g(x)}$.

Table 1

x	1	2	3	4	5
$f(x)$	-5	-2	2	4	7
$f'(x)$	5	6	2	3	3
$f''(x)$	1	-1	-3	-2	0

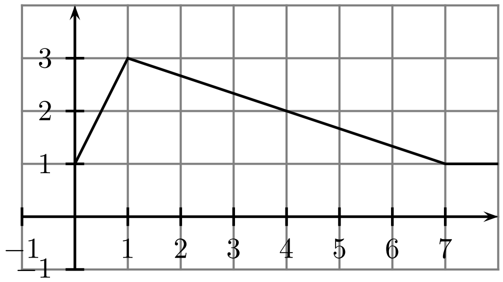


Figure 2: Graph of $g(x)$

Evaluate each of the following. **To receive partial credit you must show your work!**

(a) (4 points) $\left(f^{-1}\right)'(2)$

$$\left(f^{-1}\right)'(2) = \frac{1}{f'\big(f^{-1}(2)\big)} = \frac{1}{f'(3)} = \frac{1}{2}$$

(b) (4 points) $h'(4)$

By the chain rule, $h'(x) = f'\big(g(x)\big) \cdot g'(x)$. Therefore,

$$h'(4) = f'\big(g(4)\big) \cdot g'(4) = f'(2) \cdot \frac{-1}{3} = -2$$

(c) (4 points) $h''(4)$ [*Hint: you may want to use your work from part (b).*]

To find h'', we differentiate the formula we obtained for $h'(x)$ in part (b). Using the product rule and the chain rule, we find

$$\begin{aligned} h''(x) &= \Big(f'\big(g(x)\big)\Big)' \cdot g'(x) + f'\big(g(x)\big) \cdot g''(x) \\ &= f''\big(g(x)\big) \cdot g'(x) \cdot g'(x) + f'\big(g(x)\big) \cdot g''(x) \\ &= f''\big(g(x)\big) \cdot \big(g'(x)\big)^2 + f'\big(g(x)\big) \cdot g''(x) \end{aligned}$$

Therefore, $h''(4) = f''(2) \cdot \frac{1}{9} + f'(2) \cdot 0 = -\frac{1}{9} \approx -0.111$.

(d) (4 points) $j'(4)$

By the quotient rule,

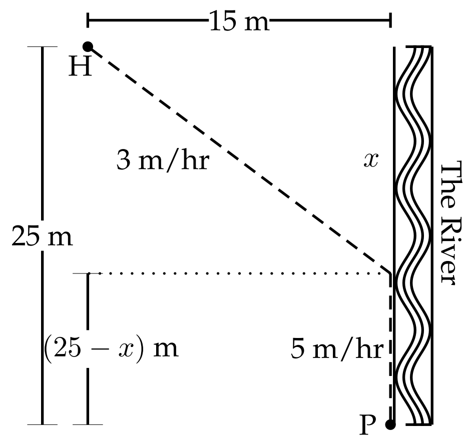
$$j'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)^2}.$$

Therefore,

$$j'(4) = \frac{g(4) \cdot f'(4) - f(4) \cdot g'(4)}{g(4)^2} = \frac{2 \cdot 3 + 4 \cdot \frac{1}{3}}{4} = \frac{11}{6} \approx 1.833.$$

4. (16 points) The Awkward Turtle is going to a dinner party! Unfortunately, he's running quite late, so he wants to take the quickest route. The Awkward Turtle lives in a grassy plain (his home is labeled H in the figure below), where his walking speed is a slow but steady 3 meters per hour. The party is taking place southeast of his home, on the bank of a river (denoted by P in the figure). The river flows south at a constant rate of 5 meters per hour, and once he gets to the river, the Awkward Turtle can jump in and float the rest of the way to the party on his back. A typical path the Awkward Turtle might take from his house to the party is indicated in the figure below by a dashed line.

   What is the shortest amount of time the entire trip (from home to dinner party) can take? [*Recall that rate* $\times$ *time = distance.*]



Let $t(x)$ denote the amount of time the trip takes if the Awkward Turtle floats along the river for $(25-x)$ meters (see the picture above). By the Pythagorean theorem, the distance the turtle will walk across the grassy plain is $\sqrt{15^2+x^2}$; therefore, we have

$$t(x) = \frac{1}{3}\sqrt{15^2+x^2} + \frac{25-x}{5}.$$

To minimize this, we take the derivative and set it equal to 0. A computation shows that

$$t'(x) = \frac{1}{3}\cdot\frac{x}{\sqrt{15^2+x^2}} - \frac{1}{5}.$$

Setting this equal to 0 and solving yields $x = \frac{45}{4} = 11.25$ m. Thus, such a trip takes $t(11.25) = 9$ hours.

Using the quotient rule we see that

$$t''(x) = \frac{1}{3}\left(\frac{\sqrt{15^2+x^2} - \frac{x^2}{\sqrt{15^2+x^2}}}{15^2+x^2}\right) = \frac{1}{3}\left(\frac{15^2}{\left(\sqrt{15^2+x^2}\right)^3}\right)$$

Since $t''(x) > 0$ for all x, we have that 9 hours is a local minimum. To determine whether $t = 9$ is the global minimum, we can either show that since there is only one critical point the local minimum is the global minimum or check the endpoints. The least the turtle can float along the river is 0 meters, in which case the trip takes $t(0) = 10$ hours; the greatest distance he can float is 25 meters, in which case the trip would take $t(25) \approx 9.718$ hours. Therefore, 9 hours is indeed the global minimum. (Alternatively, one can see this from the graph the function $y = t(x)$.)

Minimal time = <u>9 hours</u>

5. Your friend starts a small company which sells awesome t-shirts for \$10 apiece. The table below shows the cost of making different numbers of shirts:

q (number of shirts made)	5	10	15	20	25	30	35	40	45	50
$C(q)$ (cost, in \$)	100	130	150	168	184	196	206	218	236	256

(a) (2 points) Write an expression for the revenue function $R(q)$.

$R(q) = 10q$, measured in dollars.

(b) (4 points) How many shirts should your friend aim to sell, if her goal is to maximize profit? Explain.

The profit function $\pi(q) = R(q) - C(q)$. The critical points of this function are those q which make $\pi'(q) = 0$, as well as the endpoints. We check these.

From above, we know that $\pi'(q) = R'(q) - C'(q) = 10 - C'(q)$. In the table above, the largest difference between consecutive values of $C(q)$ is 30 (which is $C(10) - C(5)$), which means the largest value $C'(q)$ takes on the interval is 6. Therefore, as far as we can glean from the information given in the table, we should expect that $\pi'(q)$ is positive everywhere in the interval $0 \leq q \leq 50$. So, the maximum should be at one of the endpoints.

When $q = 5$ (i.e. 5 shirts are sold), your friend loses money, since $\pi(5) = -50$. At the other extreme, if your friend sells 50 awesome t-shirts, she will make a tidy profit of $\pi(50) = 244$ dollars. Thus, she should aim to sell 50 shirts (and probably more, if that's a possibility).

6. (6 points) The radius of a spherical balloon is increasing by 3 cm per second. At what rate is air being blown into the balloon at the moment when the radius is 9 cm? Make sure you include units! [*Hint: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.*]

If $V(t)$ denotes the volume of the balloon at some time t, the rate at which air is being blown into it is $\frac{dV}{dt}$. By chain rule, since the volume of the balloon is $V = \frac{4}{3}\pi r^3$, we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\ &= (4\pi r^2) \cdot \frac{dr}{dt} \end{aligned}$$

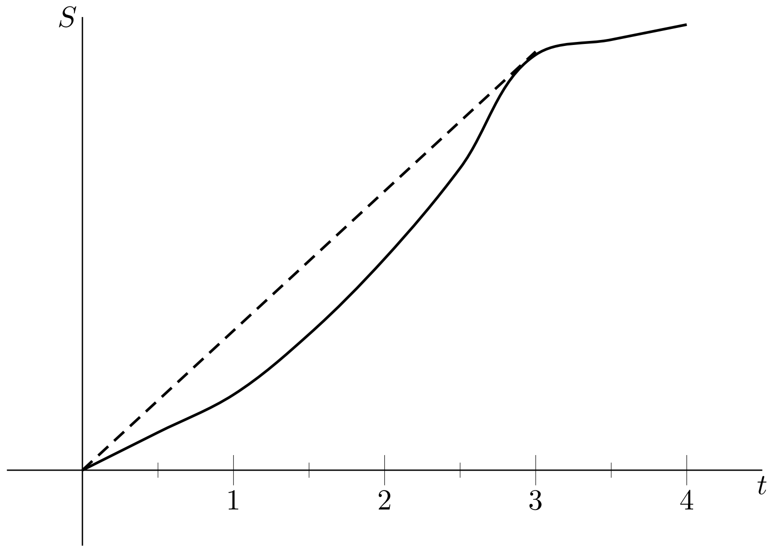
We are given in the problem statement that $\frac{dr}{dt} = 3$, whence

$$\frac{dV}{dt} = 12\pi r^2.$$

Thus when $r = 9$,

$$\frac{dV}{dt} = 972\pi \approx 3053.6 \text{ cm}^3/\text{s}.$$

7. You decide to take a weekend off and drive down to Chicago. The graph below represents your distance S from Ann Arbor, measured in miles, t hours after you set out.



Let $A(t)$ be the slope of the line connecting the origin $(0,0)$ to the point $(t, S(t))$.

(a) (3 points) What does $A(t)$ represent in everyday language?

> $A(t) = \dfrac{S(t)}{t}$ represents your average velocity, in miles per hour, during the first t hours of the trip.

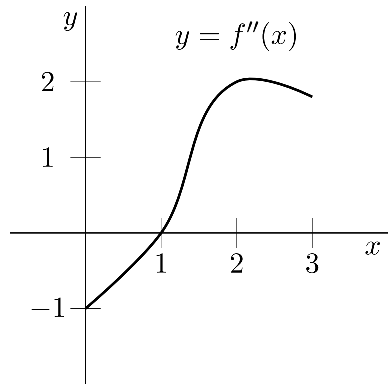
(b) (3 points) Estimate the time t at which $A(t)$ is maximized. Write a one sentence explanation and use the graph above to justify your estimate.

> $A(t)$ is maximized when the line connecting the origin to the curve is steepest. By inspecting the graph, it is clear that this happens at around $t = 3$ hours, at which point the line is tangent to the curve. (This line is indicated in the figure by the dashed line.)

(c) (4 points) Use calculus to explain why $A(t)$ has a critical point when the line connecting the origin to the point $(t, S(t))$ is tangent to the curve $S(t)$.

> By applying the quotient rule, we find that
>
> $$A'(t) = \frac{t \cdot S'(t) - S(t)}{t^2}.$$
>
> At any non-zero value of t for which the line through the origin to $(t, S(t))$ is tangent to the curve, we must have $S'(t)$ (the slope of the tangent line) equal to $S(t)/t$ (the slope of the line from the origin to $(t, S(t))$); thus, for any such t, $S(t) = t \cdot S'(t)$. Thus, for any such t, $A'(t) = 0$, i.e. $A(t)$ has a critical point there.

8. The figure below shows the graph of the *second* derivative of f, on the interval $[0, 3]$.



Assume that $f'(1) = 1$ and $f(1) = 0$.

(a) (5 points) Can $f'(x) = 0.5$ for some x in $[0, 3]$? Why or why not?

$f'(x)$ is decreasing on the interval $[0, 1]$ (since $f''(x) \leq 0$), and increasing on the interval $[1, 3]$ (since $f''(x) \geq 0$). Thus, on the whole interval $[0, 3]$, $f'(x)$ has a minimum at $x = 1$. Since $f'(1) = 1$, we deduce that $f'(x) \geq 1$ for all x in the interval $[0, 3]$. In particular, $f'(x)$ cannot equal 0.5 in the interval.

(b) (5 points) Explain why f has a global maximum at $x = 3$.

From above, $f'(x) \geq 1$ for all x in the interval $[0, 3]$. In particular, $f'(x)$ is always positive, so f is everywhere increasing on this interval. This means f attains its global maximum at the rightmost endpoint of the interval, namely, at $x = 3$.

9. (a) (4 points) Suppose that the tangent line to the function $y = f(x)$ at $x = c$ passes through the origin. Express $\left.\frac{dy}{dx}\right|_{x=c}$ in terms of c and $f(c)$.

We are given that the line passes through the origin and the point $\big(c, f(c)\big)$, so its slope is $\frac{f(c)}{c}$. On the other hand, by definition its slope is $\left.\frac{dy}{dx}\right|_{x=c}$. Therefore,

$$\left.\frac{dy}{dx}\right|_{x=c} = \frac{f(c)}{c}.$$

(b) (6 points) Consider the graph of $xy = ae^{by}$, where both a and b are positive (non-zero) constants. Determine $\frac{dy}{dx}$.

Differentiating both sides of the equation implicitly, we find

$$y + xy' = ae^{by} \cdot by'$$

Solving for y' we find

$$\frac{dy}{dx} = \frac{y}{abe^{by} - x}.$$

(c) (6 points) Write down the equations of all lines passing through the origin which are tangent to the curve $xy = ae^{by}$, where as before a and b are positive (nonzero) constants. [*Hint: You may find it helpful to rewrite your answer to 9b without exponentials, by using substitution – by the definition of the curve, you can replace the quantity* ae^{by} *by* xy.]

[Continued on next page....]

Following the hint, we rewrite the answer to (b) in the form

$$\frac{dy}{dx} = \frac{y}{bxy - x}.$$

Suppose that a line through the origin is tangent to the curve at the point (x_0, y_0). Note that neither x_0 nor y_0 can be 0; otherwise, from the equation of the curve we would have $0 = ae^{by_0}$, which would contradict the positivity of a.

By the same reasoning as in part (a),

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \frac{y_0}{x_0}.$$

On the other hand, from above,

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \frac{y_0}{bx_0y_0 - x_0}.$$

Therefore, $\frac{y_0}{x_0} = \frac{y_0}{bx_0y_0 - x_0}$. Dividing both sides of the equation by $\frac{y_0}{x_0}$ (this is OK since y_0 is not zero), we obtain

$$1 = \frac{1}{by_0 - 1}.$$

This immediately implies that $y_0 = \frac{2}{b}$, from which (using the equation defining the curve) we deduce that $x_0 = \frac{ab}{2}e^2$.

Thus, the slope of the line in question is $\frac{y_0}{x_0} = \frac{4}{ab^2e^2}$. So the equation of the line is

$$y = \frac{4}{ab^2e^2}x.$$