MATH 115 – FINAL EXAM

April 23, 2009

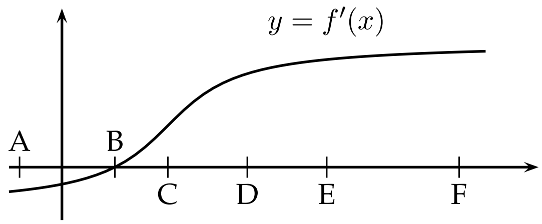
NAME: SOLUTIONS

INSTRUCTOR: ______________ SECTION NUMBER: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 10 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	14	
2	8	
3	9	
4	10	
5	8	
6	14	
7	16	
8	10	
9	11	
TOTAL	100	

1. (2 points each) Suppose f is a twice-differentiable function. Use the graph of the derivative f', shown below, to answer the following questions. No explanations are required.



(a) At which of the marked x-values does f attain a global minimum on the interval [A,F]?

B

(b) At which of the marked x-values does f attain a global maximum on the interval [A,F]?

F

(c) At which of the marked x-values does f' attain a global minimum on the interval [A,F]?

A

(d) At which of the marked x-values does f' attain a global maximum on the interval [A,F]?

F

(e) At which of the marked x-values does f'' attain a global maximum on the interval [A,F]?

C

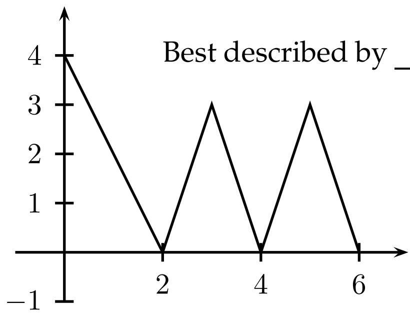
(f) For which of the marked x-values does $\int_A^x f'(t)\,dt$ attain a global minimum on the interval [A,F]?

B

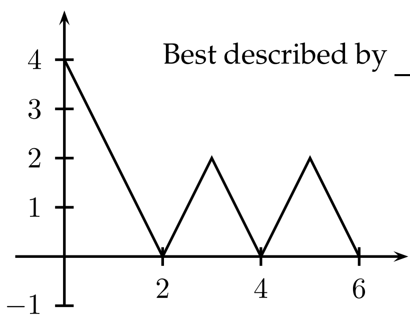
(g) For which of the marked x-values does $\int_A^x f'(t)\,dt$ attain a global maximum on the interval [A,F]?

F

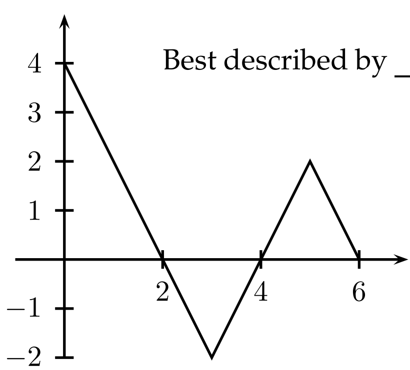
2. (2 points each) Next to each of the functions graphed on the left below, identify which one of the inequalities on the right below best describes the situation. Here, L is the left Riemann sum for $\int_0^6 f(x)\,dx$ using three equal subdivisions, and R is the right Riemann sum using three equal subdivisions. [*You may find it helpful to compute L, R, and the integral for each graph.*]



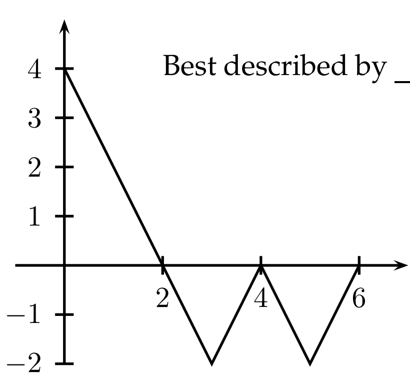
Best described by **(f)**



Best described by **(g)**



Best described by **(h)**



Best described by **(i)**

(a) $L < R < \int_0^6 f(x)\,dx$

(b) $L = R < \int_0^6 f(x)\,dx$

(c) $L < R = \int_0^6 f(x)\,dx$

(d) $L < \int_0^6 f(x)\,dx < R$

(e) $L = \int_0^6 f(x)\,dx < R$

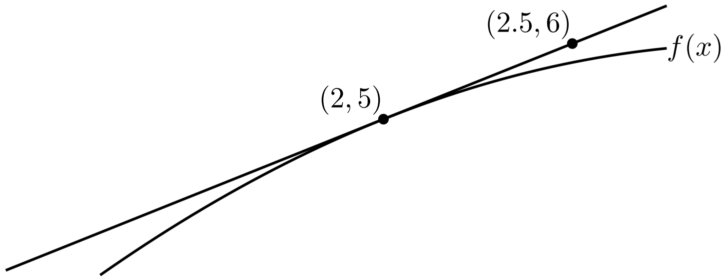
(f) $R < L < \int_0^6 f(x)\,dx$

(g) $R < L = \int_0^6 f(x)\,dx$

(h) $R < \int_0^6 f(x)\,dx < L$

(i) $R = \int_0^6 f(x)\,dx < L$

3. The figure below shows a differentiable function f and the line tangent to the graph at the point $(2,5)$: (*picture not drawn to scale*)



(a) (3 points) Approximate $f(2.01)$. Is your approximation an over or underestimate? Explain.

¿From the two points given, the slope of the tangent line is 2. Therefore $f'(2) = 2$ and

$$f(2.01) \approx f(2) + 0.01 \cdot f'(2) = 5.02$$

Since f is concave down this must be an overestimate.

(b) (3 points) Evaluate $h'(2)$ if $h(x) = \Big(f(x)\Big)^3$.

By the chain rule,

$$h'(x) = 3\Big(f(x)\Big)^2 f'(x)$$

whence

$$h'(2) = 3\Big(f(2)\Big)^2 f'(2) = 150$$

(c) (3 points) Evaluate $g'(2)$ if $g(x) = e^{f(x)}$.

By the chain rule,

$$g'(x) = f'(x)\, e^{f(x)}$$

whence

$$g'(2) = f'(2)\, e^{f(2)} = 2\, e^5 \approx 296.826$$

4. (10 points) A car initially traveling 80 ft / sec brakes to a stop in 8 seconds. Its velocity is recorded every 2 seconds and is given in the following table:

t (seconds)	0	2	4	6	8
$v(t)$ (ft/sec)	80	52	28	10	0

(a) Give a good estimate for the distance the car traveled during the course of the 8 seconds. Is your approximation an over or underestimate? How do you know?

Type of sum	**Evaluation**	**Over or underestimate?**
Left sum	$(80)(2) + (52)(2) + (28)(2) + (10)(2) = 340$ ft	Over: velocity is decreasing
Right sum	$(52)(2) + (28)(2) + (10)(2) + (0)(2) = 180$ ft	Under: velocity is decreasing
Average	260 ft	Over: velocity is concave up

(b) To estimate the distance traveled accurate to within 20 feet, how often should the velocity be recorded?

Suppose we record every Δt seconds. Since the velocity is decreasing, the right Riemann sum must be smaller than the distance traveled, which in turn must be smaller than the left Riemann sum. We have

$$L - R = \Big(v(0) - v(8)\Big)\Delta t = 80\Delta t.$$

Therefore if we measure the velocity every $\Delta t = 0.25$ seconds, the left Riemann sum L will be within 20 ft of the actual distance traveled.

(c) Approximate the acceleration of the car 4 seconds after the brakes were applied.

We can approximate this as either $\dfrac{v(4) - v(2)}{4 - 2} = -12$ ft/s^2, $\dfrac{v(6) - v(4)}{6 - 4} = -9$ ft/s^2, or as the average of the two (-10.5 ft/s^2).

5. (8 points) A potter has a fixed volume of clay in the form of a cylinder. As he rolls the clay, the length of the cylinder, L, of increases, while the radius, r, decreases. If the length of the cylinder is increasing at a constant rate of 0.2 cm per second, find the rate at which the radius is changing when the radius is 1.5 cm and the length is 4 cm.

[*Recall that the volume of a cylinder of radius r and length L is $\pi r^2 L$.*]

Differentiating the formula for the volume, we find

$$\frac{dV}{dt} = \pi\, 2r \frac{dr}{dt}\, L + \pi\, r^2 \frac{dL}{dt}.$$

Although the shape of the clay is changing, the volume is not, so $\dfrac{dV}{dt} = 0$. Combining these two statements,

$$\pi\, 2r \frac{dr}{dt}\, L + \pi\, r^2 \frac{dL}{dt} = 0$$

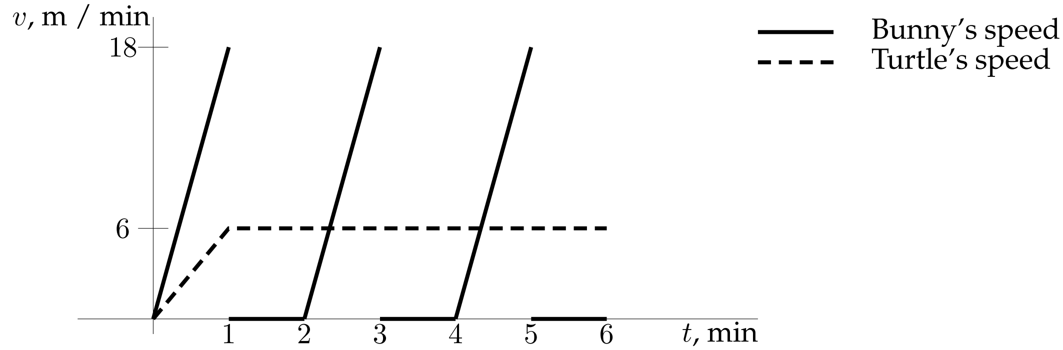
whence

$$\frac{dr}{dt} = -\frac{r\, \frac{dL}{dt}}{2L}.$$

Plugging in the given values $L = 4$, $r = 1.5$, $\dfrac{dL}{dt} = 0.2$ we deduce

$$\frac{dr}{dt} = -0.0375 \text{ cm / s}.$$

6. The Awkward Turtle is competing in a race! Unfortunately his archnemesis, the Playful Bunny, is also in the running. The two employ very different approaches: the Awkward Turtle takes the first minute to accelerate to a slow and steady pace which he maintains through the remainder of the race, while the Playful Bunny spends the first minute accelerating to faster and faster speeds until she's exhausted and has to stop and rest for a minute - and then she repeats this process until the race is over. The graph below shows their speeds (in meters per minute), t minutes into the race. (Assume that the pattern shown continues for the duration of the race.)



(a) (6 points) What is the Awkward Turtle's average speed over the first two minutes of the race? What is the Playful Bunny's?

> Let $T(t)$ denote the Awkward Turtle's velocity t minutes into the race, and $B(t)$ the Playful Bunny's. Then the Awkward Turtle's average velocity over the first two minutes is
>
> $$\frac{1}{2}\int_0^2 T(t)\,dt = \frac{1}{2}\left(\int_0^1 T(t)\,dt + \int_1^2 T(t)\,dt\right) = \frac{1}{2}(3+6) = 4.5 \text{ m / min}$$
>
> Similarly, the Playful Bunny's average velocity over the first two minutes is
>
> $$\frac{1}{2}\int_0^2 B(t)\,dt = \frac{1}{2}\left(\int_0^1 B(t)\,dt + \int_1^2 B(t)\,dt\right) = \frac{1}{2}(9+0) = 4.5 \text{ m / min}$$

(b) (3 points) The Playful Bunny immediately gets ahead of the Awkward Turtle at the start of the race. How many minutes into the race does the Awkward Turtle catch up to the Playful Bunny for the first time? Justify your answer.

> From part (a), we know that at two minutes into the race, the Awkward Turtle and the Playful Bunny have run the exact same distance. So, it remains to determine whether this is the first time the Turtle catches up. Since the Bunny is running faster than the Turtle the entire first minute of the race, it's clear that the Bunny is ahead that whole time, so the Turtle couldn't have caught up during the first minute. During the second minute of the race, the Bunny is standing still, so the first time the Turtle catches up must be precisely two minutes into the race.

(c) (5 points) If the race is 60 meters total, who wins? Justify your answer.

The distance the Turtle has run after x minutes is

$$\int_0^x T(t)\,dt = \int_0^1 T(t)\,dt + \int_1^x T(t)\,dt = 3 + 6(x-1) = 6x - 3 \text{ meters.}$$

Therefore it takes the Awkward Turtle 10.5 minutes to finish the race. By contrast, after 10.5 minutes the Playful Bunny has run

$$\begin{aligned}\int_0^{10.5} B(t)\,dt &= \int_0^1 B(t)\,dt + \int_2^3 B(t)\,dt + \cdots + \int_8^9 B(t)\,dt + \int_{10}^{10.5} B(t)\,dt \\ &= (5)(9) + 2.25 \\ &= 47.25 \text{ meters.}\end{aligned}$$

So the Awkward Turtle wins!

7. (4 points each) Table 1 below displays some values of an invertible, twice-differentiable function $f(x)$, while Figure 2 depicts the graph of the function $g(x)$.

Table 1

x	1	2	3	4	5
$f(x)$	-5	-2	2	4	7
$f'(x)$	5	6	2	3	3
$f''(x)$	1	-1	-3	-2	0

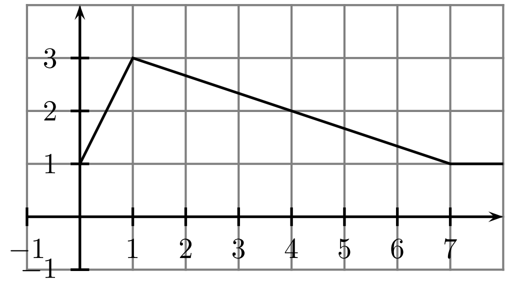


Figure 2: Graph of $g(x)$

Evaluate each of the following. Show your work.

(a) $\displaystyle\int_0^7 g(x)\,dx$

The region under g from 0 to 7 can be viewed as a 1×7 rectangle with a triangle sitting on top of it with base 7 and height 2. Thus,

$$\int_0^7 g(x)\,dx = 7 + \left(\frac{1}{2}\right)(7)(2) = 14$$

(b) $\displaystyle\int_1^3 f'(x)\,dx$

By the fundamental theorem of calculus,

$$\int_1^3 f'(x)\,dx = f(3) - f(1) = 7$$

(c) $\displaystyle\int_1^5 \Big(3f''(x) + 4\Big)\,dx$

$$\begin{aligned}\int_1^5 \Big(3f''(x) + 4\Big)\,dx &= 3\int_1^5 f''(x)\,dx + \int_1^5 4\,dx \\ &= 3\Big(f'(5) - f'(1)\Big) + 4(5-1) \\ &= 10\end{aligned}$$

(d) $\displaystyle\int_1^4 \Big(f'(x)g(x) + f(x)g'(x)\Big)\,dx$

Recognizing $f'(x)g(x) + f(x)g'(x)$ as the derivative of the $f(x)g(x)$ (the product rule!), we apply the fundamental theorem of calculus to find

$$\int_1^4 \Big(f'(x)g(x) + f(x)g'(x)\Big)\,dx = f(x)g(x)\Big|_1^4 = (4)(2) - (-5)(3) = 23$$

8. (10 points) A typical student spends the 24 hours leading up to this exam sleeping, studying, eating, and Facebook stalking. Suppose the total amount of time spent on eating and Facebook is 8 hours. The student's score on the exam, E (out a possible 100 points) depends on S, the number of hours of sleep the student enjoys during the 24 hours leading up to the exam. To be precise,

$$E(S) = 40 \sin\left(\frac{5\pi}{51}(S - 3.4)\right) + 36$$

How many hours should the student study in the day leading up to the exam to maximize his / her score?

[*You must use calculus - not just your calculator - and show your work to receive full credit.*]

We are looking for the global maximum of E. Thus start by identifying which values of S make $E'(S) = 0$:

$$E'(S) = (40)\left(\frac{5\pi}{51}\right)\cos\left(\frac{5\pi}{51}(S - 3.4)\right) = 0$$

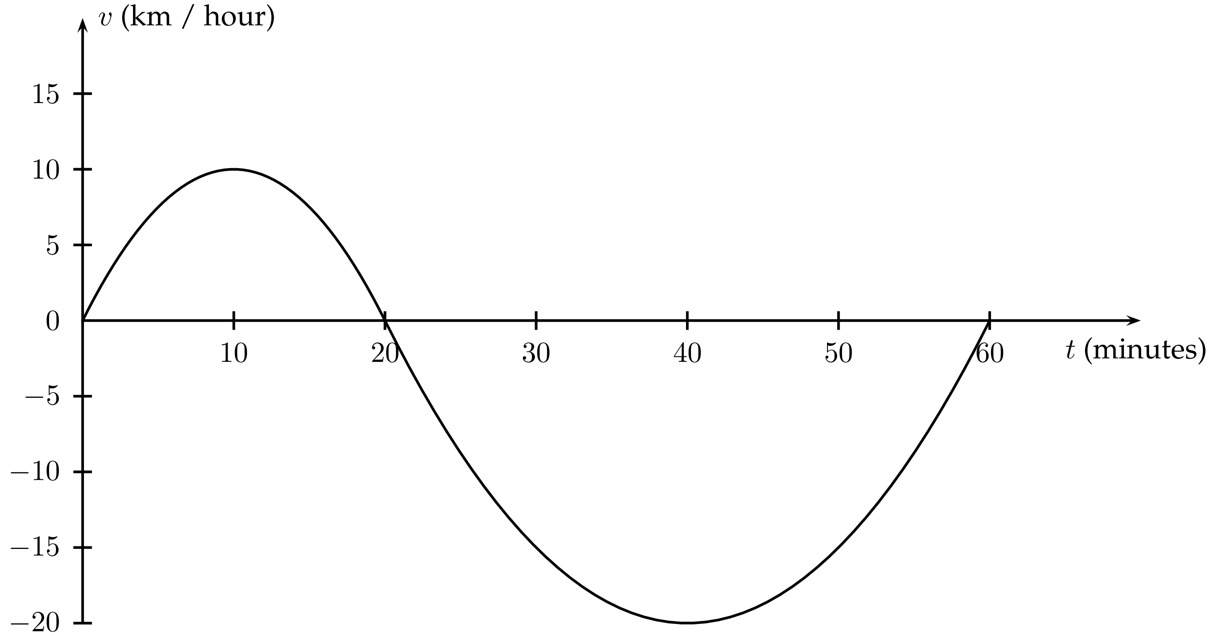
implies

$$\frac{5\pi}{51}(S - 3.4) = \frac{\pi}{2} + k\pi$$

for some integer k. It follows that any solution to this is of the form $S = 8.5 + \frac{51k}{5}$ for k an integer. Since taking any negative value of k gives a negative value of S, and taking any positive value of k gives a value of S larger than 16, the only solution to $E'(S) = 0$ is $S = 8.5$ hours. Also, since $E'(S) > 0$ for S slightly below 8.5 and $E'(S) < 0$ for S slightly larger than 8.5, E must attain a maximum at $S = 8.5$.

To check that this is the global maximum on the interval, it suffices to test the the endpoints. Plugging these in, we find $E(0) \approx 1.359$ and $E(16) \approx 9.052$, both of which are smaller than $E(8.5) = 76$. So, the student's exam score is maximized when he / she sleeps 8.5 hours; this means the student must study $16 - 8.5 = 7.5$ hours to maximize his / her score.

9. A bicyclist is pedaling along a straight road for one hour with a velocity v shown in the figure below. She starts out five kilometers from a lake; positive velocities take her toward the lake.



(a) (2 points) Does the cyclist ever turn around? If so, at what time(s)?

The cyclist turns around 20 minutes into the ride.

(b) (3 points) When is she going the fastest? How fast is she going then? Is she going toward or away from the lake?

40 minutes into the ride, the cyclist is riding away from the lake at a rate of 20 km / hr.

(c) (3 points) When is she closest to the lake? Approximately how close to the lake does she get?

20 minutes into the ride, the cyclist has gone

$$\int_0^{\frac{1}{3}} v(t)\,dt \approx \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)(10) = \frac{5}{3} \text{ km}$$

from her starting position. Since she was traveling towards the lake this whole time, and she started 5 km away from it, she is now about $5 - 1.667 = 3.333$ km from the lake.

(d) (3 points) When is she farthest from the lake? Approximately how far from the lake is she then?

The cyclist is furthest from the lake 1 hour into her ride. From her position 20 minutes into the ride, her position has changed by

$$\int_{\frac{1}{3}}^{1} v(t)\,dt \approx \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(-20) = -\frac{20}{3} \text{ km.}$$

In part (c) we approximated she was 3.333 km from the lake 20 minutes into the ride; thus she is approximately 6.667 km further away from the lake at the end of the ride, i.e. she is approximately 10 km away at the end of her ride.