Math 115 — First Midterm

February 9, 2010

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. **Use the techniques of calculus to solve the problems on this exam.**

Problem	Points	Score
1	13	
2	8	
3	6	
4	10	
5	10	
6	12	
7	8	
8	10	
9	15	
10	8	
Total	100	

1. [13 points] For each problem below, circle **ALL** of the statements that **MUST** be true. (The four parts (a)-(d) are independent of each other. No explanations are required.)

a. [3 points] Suppose f is a differentiable function which is concave up on its entire domain, $(-\infty, \infty)$.

- ○ $\boxed{\lim_{x \to 1} f(x) = f(1)}$
- ○ $f(2) \geq f(1)$
- ○ $\boxed{f'(2) \geq f'(1)}$

b. [3 points] Suppose that $h(t)$ gives the height of a ball, measured in feet above ground level, t seconds after it is thrown off a bridge. Assume that the derivative of h is given by the formula $h'(t) = -32t + 64$.

- ○ $\boxed{\text{The ball reaches its maximum height 2 seconds after being thrown.}}$
- ○ The ball reaches a maximum height of 64 feet from the ground.
- ○ The bridge is 64 feet off the ground.

c. [4 points] Suppose that A and B are positive constants and $A < B$.

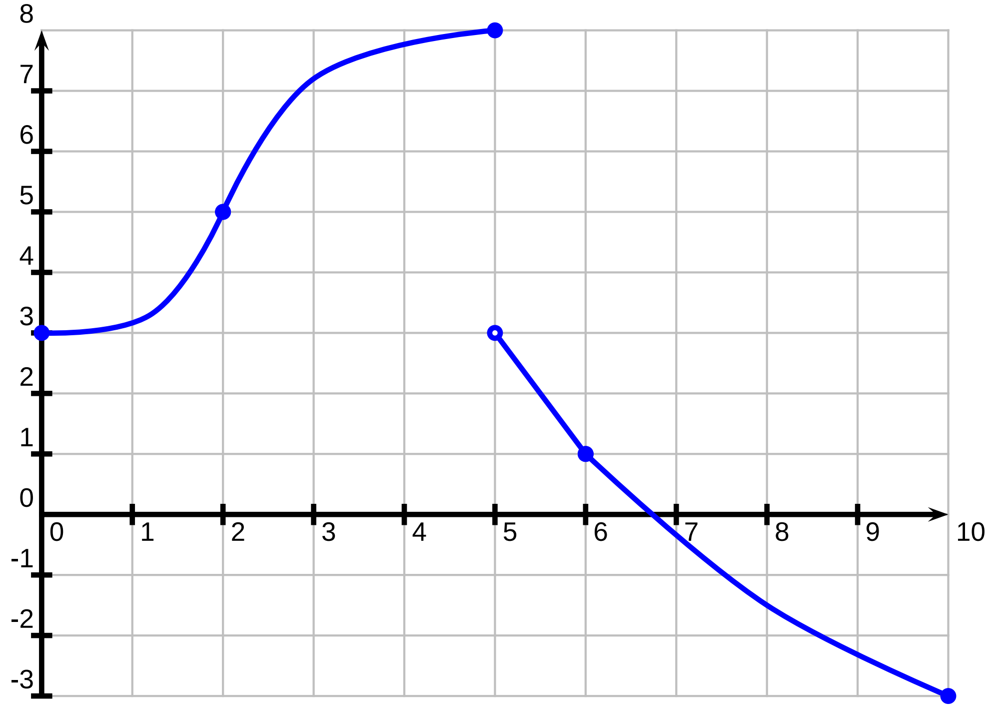
- ○ $(\ln e^A)(\ln e^B) = A + B$
- ○ $\boxed{\ln(10^{-A}) < 0}$
- ○ $\ln(A^2 + B) = 2 \ln A + \ln B$
- ○ $\boxed{\log A < \log B}$

d. [3 points] Suppose that $f(x) = -Ae^{-Bx}$ for some positive constants A and B.

- ○ $\boxed{f'(x) > 0 \text{ for all } x}$
- ○ f' is increasing
- ○ $\boxed{f \text{ is increasing}}$

2. [8 points] On the axes provided below, sketch the graph of a function f, defined on the interval $[0, 10]$, which satisfies **ALL** of the following properties. (Hint: the function f is not required to be continuous.)

- f is invertible on the entire domain $[0, 10]$.
- $f(0) = 3$
- $f^{-1}(5) = 2$
- $f'(x) > 0$ for $0 < x < 5$.
- $f''(x) > 0$ for $0 < x < 2$.
- $f'(x)$ is decreasing on the interval $(2, 5)$.
- $f'(x) = -2$ for $5 < x < 6$.
- $\lim_{x \to 10^-} f(x) = -3$.



3. [6 points] Consider the function $f(x) = 13x \sin(x^2 + 1)$. Write down the limit definition of $f'(2)$. (You do not need to estimate or compute the derivative.)

Solution:

$$f'(2) = \lim_{h \to 0} \frac{13(2+h)\sin((2+h)^2+1) - 26\sin(5)}{h}$$

OR

$$f'(2) = \lim_{h \to 0} \frac{13(2+h)\sin((2+h)^2+1) - 13(2)\sin(2^2+1)}{h}$$

4. [10 points] Let $W = f(t)$ be the amount of water, in gallons, in a bathtub at time t, in minutes. Suppose that Anna turns on the water in the bathtub at time $t = 0$. After exactly 6 minutes, Anna turns off the water and proceeds to bathe her new puppy, Asta. Asta loves the water and doesn't splash or fuss, so Anna has the opportunity to give her a good shampoo. After a 10 minute bath, Anna pulls the plug, and the bathtub takes exactly 4.5 minutes to empty. Because Asta sheds, the water drains quickly at first and then the draining slows down. Anna keeps clearing the hair to assist the draining.

a. [3 points] For which values of t between 0 and 25 is the quantity $\frac{dW}{dt}$ ...

positive?

Solution: $\frac{dW}{dt}$ is positive for $0 < t < 6$

zero?

Solution: $\frac{dW}{dt}$ is zero for $6 < t < 16$ and for $20.5 < t < 25$

negative?

Solution: $\frac{dW}{dt}$ is negative for $16 < t < 20.5$

b. [2 points] Suppose that the line tangent to the graph of $W = f(t)$ at $t = 18$ passes through the points $(17, 17.5)$ and $(18.5, 4)$. Find $f'(18)$ and include units with your answer.

Solution: $f'(18) = \dfrac{4 - 17.5}{18.5 - 17} = -9$ gallons per minute

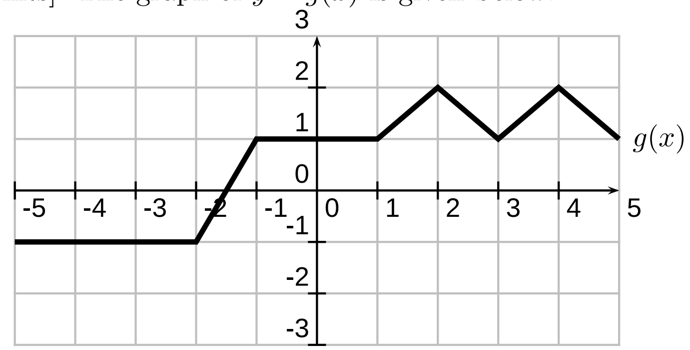
c. [2 points] Find $f(18)$ and include units with your answer.

Solution: Since the slope at $t = 18$ is -9 and the point $(18, f(18))$ is on the tangent line, we can subtract 9 from $f(17)$. Thus, $f(18) = 8.5$ gallons.

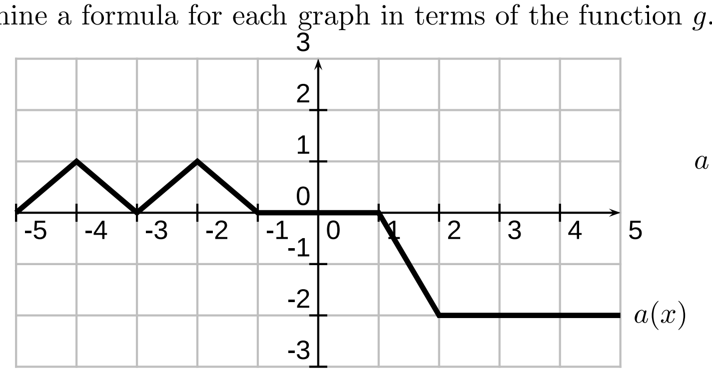
d. [3 points] Use your answers from (b) and (c) to estimate $f(20)$. Explain, in practical terms, why this estimate is or is not reasonable.

Solution: Using the information from (b) and (c), the estimate for $f(20) \approx 8.5 + 2(-9) = -9.5$ gallons. Clearly this estimate is unreasonable, since there cannot be a negative number of gallons in the bathtub. Alternate explanation: we know $f(20)$ (the number of gallons in the tub after 20 minutes) must be positive, since the bathtub doesn't finish emptying until $t = 20.5$.

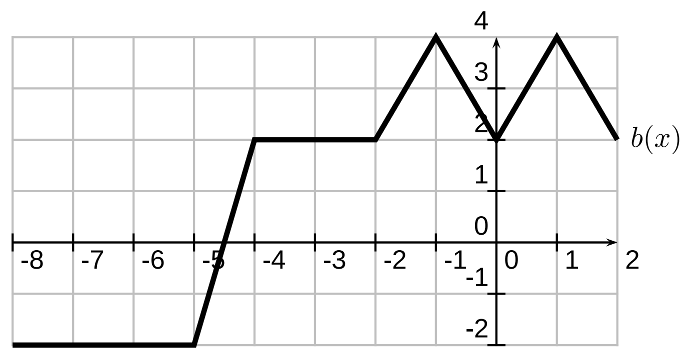
5. [10 points] The graph of $y = g(x)$ is given below.



a. [6 points] The graphs of the following two functions are related to the graph of g. Determine a formula for each graph in terms of the function g.



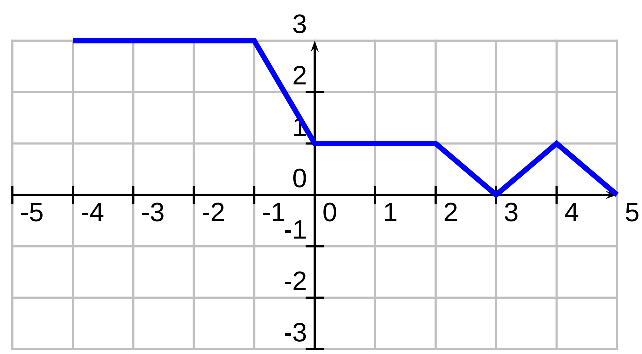
$a(x) =$ $g(-x) - 1$



$b(x) =$ $2g(x+3)$

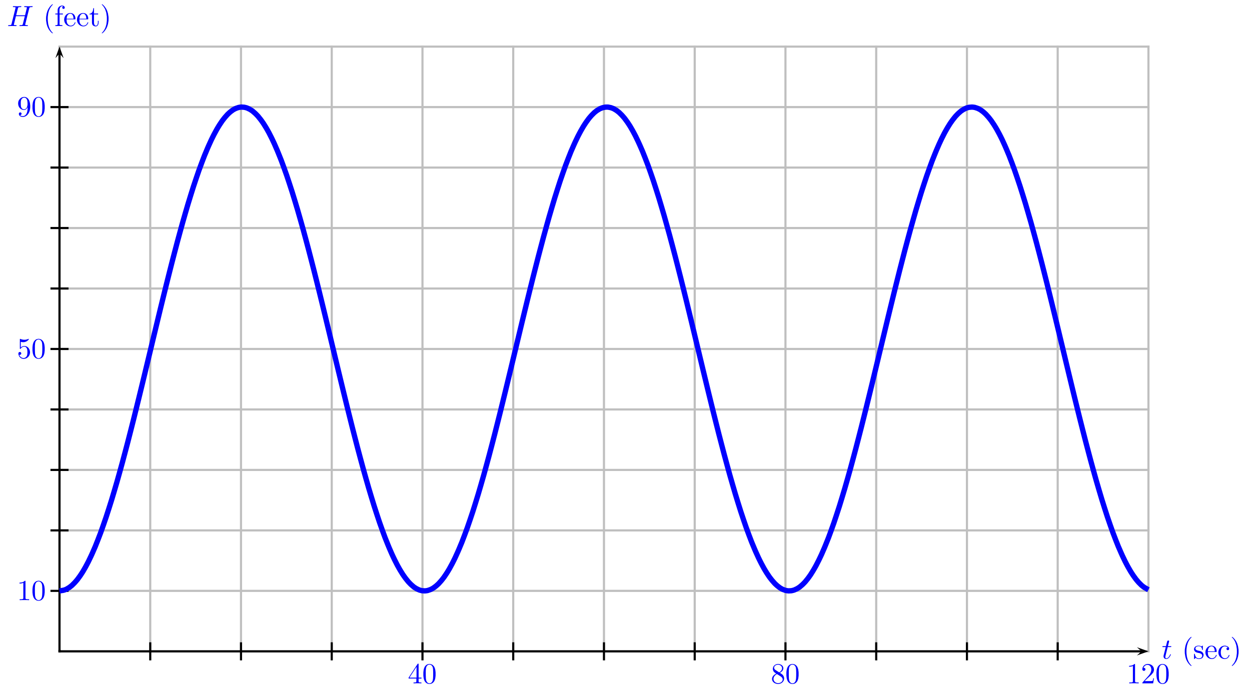
b. [4 points] Carefully sketch as much of the function $c(x)$ as will fit on the axes below, where

$$c(x) = -g(x-1) + 2.$$



6. [12 points] At the county fair, there is a ferris wheel with radius 40 feet. Riders board at the lowest point of the ferris wheel, from a platform 10 feet off the ground. Once the ride begins, the ferris wheel completes 3 revolutions in 120 seconds. Suppose that you are the last rider to board (so you begin at the lowest point), and the function $H(t)$ measures your height off the ground (in feet), t seconds after the ride starts.

a. [4 points] On the grid below, sketch a graph $H(t)$ for one complete ride (3 revolutions). Be sure to carefully label the axes.



b. [4 points] Find the period and amplitude of $H(t)$.

Solution: 40 seconds

period = ______________________

Solution: 40 feet

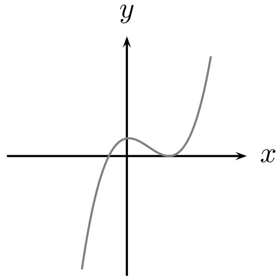
amplitude = ______________________

c. [4 points] Find a formula for $H(t)$.

Solution:

$$H(t) = 50 - 40\cos\left(\frac{\pi}{20}t\right)$$

7. [8 points] For each of the graphs below, select the formula beneath the graph which best fits the behavior of the graph. In each case, assume that A, B, C, D, E, F, and G are positive constants. (Circle your choice. No work or explanation is necessary.)

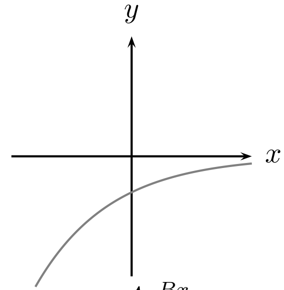


$y = A(x-B)(x+C)$

$\boxed{y = A(x-B)^2(x+C)}$

$y = -A(x+B)^2(x-C)$

$y = A(x+B)^2(x-C)$

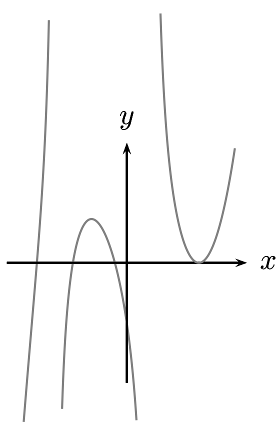


$y = Ae^{Bx}$

$y = Ae^{-Bx}$

$y = -Ae^{Bx}$

$\boxed{y = -Ae^{-Bx}}$

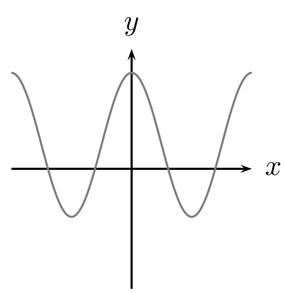


$$y = \frac{A(x-B)(x-C)(x-D)(x+E)^2}{(x+F)(x-G)}$$

$$y = \frac{A(x+B)(x+C)(x+D)(x-E)}{(x+F)(x-G)}$$

$$\boxed{y = \frac{A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)}}$$

$$y = \frac{-A(x+B)(x+C)(x+D)(x-E)^2}{(x+F)(x-G)^2}$$



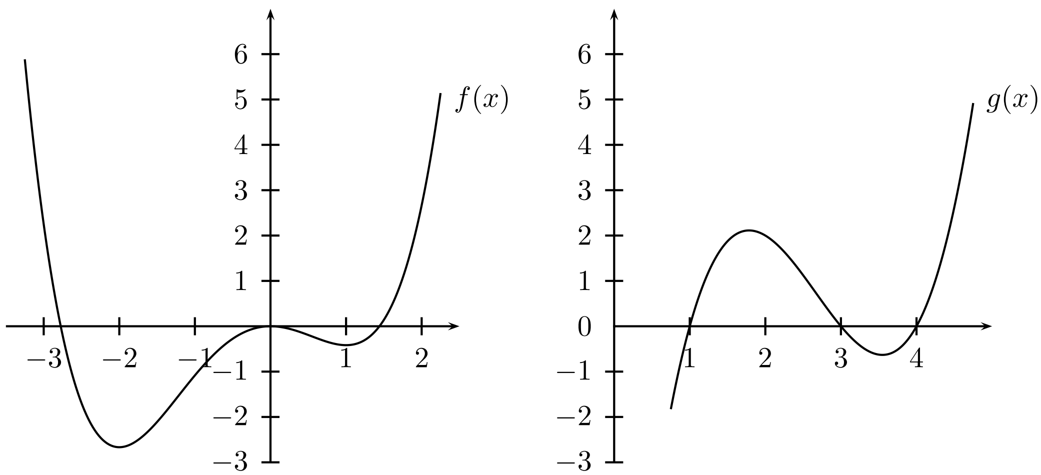
$y = -B\cos(Cx) - A$

$\boxed{y = A + B\cos(Cx)}$

$y = -A + B\sin(Cx + D)$

$y = A - B\sin(Cx)$

8. [10 points] The graphs of two functions f and g are shown below, along with a table of values for a function h.



x	-3	-2	-1	0	1	2	3
$h(x)$	15	2	-5	-6	-1	10	27

a. [4 points] Compute each of the following.

- $h(g(1)) =$ ________________________

Solution: $h(g(1)) = h(0) = -6$

- $f(1 + h(1)) =$ ________________________

Solution: $f(1 + h(1)) = f(1 + (-1)) = f(0) = 0$

b. [3 points] There exists a number B so that $f'(x) = g(x + B)$. Find B.

Solution: Since f is flat at $x = -2$, $x = 0$, and $x = 1$, we know f' has zeroes at these spots. Since g has zeroes at $x = 1$, $x = 3$, and $x = 4$, we need to shift g to the left by 3 to get f'. Thus, $B = 3$.

c. [3 points] Is it possible that $f'' = h$? Briefly justify your answer.

Solution: No. At $x = 1$, f is concave up, so $f''(1) \geq 0$, but $h(1) = -1$.

9. [15 points] Suppose that $W(h)$ is an invertible function which tells us how many gallons of water an oak tree of height h feet uses on a hot summer day.

a. [9 points] Give practical interpretations for each of the following quantities or statements.

- $W(50)$

Solution: The expression $W(50)$ represents how many gallons of water a 50 foot tall oak tree uses on a hot summer day.

- $W^{-1}(40)$

Solution: The expression $W^{-1}(40)$ represents the height of an oak tree (in feet) which uses 40 gallons of water on a hot summer day.

- $W'(5) = 3$

Solution: An oak tree which is 6 feet tall uses approximately 3 more gallons of water per hot summer day than a 5 foot tall oak tree does.
OR
If a 5 foot tall oak tree grew an extra foot, it would use approximately 3 more gallons of water per hot summer day.

b. [6 points] Suppose that an average oak tree is A feet tall and uses G gallons of water on a hot summer day. Answer each of the questions below **in terms of the function** W. You may also use the constants A and/or G in your answers.

- A farmer has a grove with 25 oak trees, and each one is 10 feet taller than an average oak tree. How much water will be used by his trees during a hot summer day?

Solution: $25W(A + 10)$ gallons

- The farmer also has some oak trees which each use 5 fewer gallons of water on a hot summer day than an average oak tree does. How tall is one of these trees?

Solution: $W^{-1}(G - 5)$ feet

10. [8 points]
According to US Census Data, the population of the city of Detroit has been declining since 1950. Suppose that $P = f(t)$ is the population of the city of Detroit (in millions of people) t years after 1950. The table below gives some values of $P = f(t)$.

t	0	10	20	30	40	50
P	1.8496	1.6701	1.5115	1.2033	1.0280	0.9513

a. [4 points]
Use the table to estimate the derivative of $f^{-1}(P)$ at $P = 1.61$. Be sure to include units with your answer.

Solution: The points $(10, 1.6701)$ and $(20, 1.5115)$ are the closest to $P = 1.61$, so we will use these for our estimate. Slopes for f^{-1} are of the form $\frac{\Delta t}{\Delta P}$ (since P is the input of f^{-1} and t is the output). Between our two points, we have

$$\frac{\Delta t}{\Delta P} = \frac{20 - 10}{1.5115 - 1.6701} \approx -63.05,$$

so the derivative of $f^{-1}(P)$ at $P = 1.61$ is approximately -63.05 years per million people.

b. [4 points]
Suppose Detroit's population decays exponentially starting in 1990. In what year will Detroit have a population of 650,000 people?

Solution: We have two points in the period starting in 1990, namely $(40, 1.0280)$ and $(50, 0.9513)$. We will use these to find the exponential decay rate. In ten years, the population multiplies by $\frac{0.9513}{1.0280}$, so in one year, the population multiplies by $\left(\frac{0.9513}{1.0280}\right)^{\frac{1}{10}}$. If we let N be the number of years since 1990, then a formula for P after 1990 is given by

$$P = 1.0280\left(\left(\frac{0.9513}{1.0280}\right)^{\frac{1}{10}}\right)^N \approx 1.0280(0.992276)^N.$$

Set $P = 0.65$ and solve for N using logs.

$$0.65 = 1.0280\left(\left(\frac{0.9513}{1.0280}\right)^{\frac{1}{10}}\right)^N.$$

$$\ln\left(\frac{0.65}{1.0280}\right) = N\ln\left(\left(\frac{0.9513}{1.0280}\right)^{\frac{1}{10}}\right).$$

$$\frac{\ln\left(\frac{0.65}{1.0280}\right)}{\ln\left(\left(\frac{0.9513}{1.0280}\right)^{\frac{1}{10}}\right)} = N \approx 59.11.$$

Thus, if the population is decaying exponentially, there will be 650,000 people in Detroit in the year 2049, or 59.11 years after 1990.