Math 115 — Second Midterm

March 25, 2010

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. **Use the techniques of calculus to solve the problems on this exam.**

Problem	Points	Score
1	12	
2	12	
3	10	
4	12	
5	13	
6	12	
7	16	
8	13	
Total	100	

1. [12 points]
For the following statements, select True if the statement is *ALWAYS* true, and select False otherwise. No explanations are required.

a. [2 points] Suppose that f is a function whose second derivative is both continuous and positive everywhere. Then

$$f(2 + \Delta x) > f(2) + f'(2)\Delta x.$$

$\boxed{\text{True}}$ False

b. [2 points] Suppose that g is a continuous function and g' is defined for all x. Then g'' is also defined for all x.

True $\boxed{\text{False}}$

c. [2 points] If a continuous function H has exactly one local maximum and two local minima, then there are exactly three distinct values of x such that $H'(x) = 0$.

True $\boxed{\text{False}}$

d. [2 points] Suppose that A and B are two continuous functions such that $A'(x) \leq B'(x)$ for all x. Then $A(x) \leq B(x)$ for all x.

True $\boxed{\text{False}}$

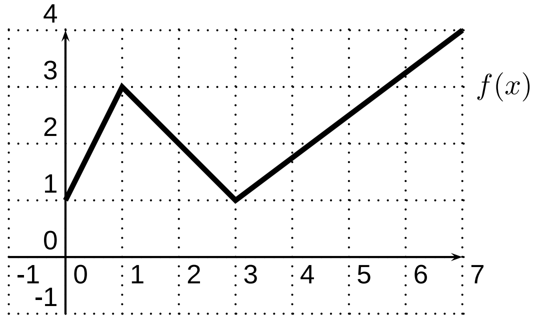
e. [2 points] Suppose $P(x)$ is a continuous function satisfying $P'(x) \geq 0$ whenever $x > 0$. Then $P(a) \leq P(b)$ whenever $0 < a < b$.

$\boxed{\text{True}}$ False

f. [2 points] If the functions R and S are inverses of each other, then R' and S' are inverses of each other.

True $\boxed{\text{False}}$

2. [12 points]
Use the graph of the function f and the table of values for the function g to answer the questions below.



x	1	2	3	4	5	6
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79
g"(x)	-2	-8	-14	-20	-26	-32

a. [6 points] Let $h(x) = \frac{g(x)}{f(2x+3)}$. Find $h'(1)$ or explain why it does not exist.

Solution: Using the quotient rule and the chain rule, we get

$$\begin{aligned} h'(x) &= \frac{g'(x)f(2x+3) - g(x)f'(2x+3)\cdot 2}{(f(2x+3))^2} \\ h'(1) &= \frac{g'(1)f(5) - g(1)f'(5)\cdot 2}{(f(5))^2} \\ &= \frac{6\cdot 2.5 - 0\cdot 0.75\cdot 2}{(2.5)^2} \\ &= \frac{6}{2.5} = \frac{12}{5} = 2.4 \end{aligned}$$

b. [6 points] Let $k(x) = g(g(x))$. Determine whether k is increasing or decreasing at $x = 2$.

Solution: Using the chain rule, we get

$$\begin{aligned} k'(x) &= g'(g(x))\cdot g'(x) \\ k'(2) &= g'(g(2))\cdot g'(2) \\ &= g'(4)\cdot g'(2) \\ &= (-27)\cdot 1 = -27 \end{aligned}$$

Since $k'(2) < 0$, we know that $k(x)$ is decreasing at $x = 2$.

3. [10 points]
Find the x- and y-coordinates of all local minima, local maxima, and inflection points of the function $f(x)$ defined below. Your answers may involve the positive constant B. You must clearly mark your answers and provide justification to receive credit.

$$f(x) = e^{-18x^2+B}$$

Solution: We will need both $f'(x)$ (chain rule) and $f''(x)$ (product rule and chain rule).

$$\begin{aligned} f'(x) &= e^{-18x^2+B}(-36x) = -36xe^{-18x^2+B} \\ f''(x) &= -36e^{-18x^2+B} + (-36x)e^{-18x^2+B}(-36x) \\ &= -36e^{-18x^2+B}(1-36x^2) \end{aligned}$$

First we will find the critical points. We know f' is never undefined, and $f'(x) = 0$ only when $x = 0$, since e^{-18x^2+B} is always positive. The y-value for $x = 0$ is $e^{-18\cdot 0^2+B} = e^B$ Thus, $(0, e^B)$ is the only critical point. Since $f''(0) = -36e^B(1) = -36e^B$ is negative, we know this point is a local maximum.

LOCAL MAXIMUM AT $(0, e^B)$.

Next we will find potential inflection points. We know f'' is never undefined, and $f''(x) = 0$ when $1 - 36x^2 = 0$, since e^{-18x^2+B} is always positive. Solving, we find that $f''(x) = 0$ when $x = \pm\frac{1}{6}$. Both of these points have a y-value of $e^{-18(\frac{1}{6})^2+B} = e^{-\frac{1}{2}+B}$. We need to test f'' near these x-values to check whether we actually have inflection points.
When $x < -\frac{1}{6}$, $f''(x) > 0$.
When $-\frac{1}{6} < x < \frac{1}{6}$, $f''(x) < 0$.
When $x > \frac{1}{6}$, $f''(x) > 0$.
Since f'' changes sign at both of these points, f changes concavity at both points, so both are inflection points.

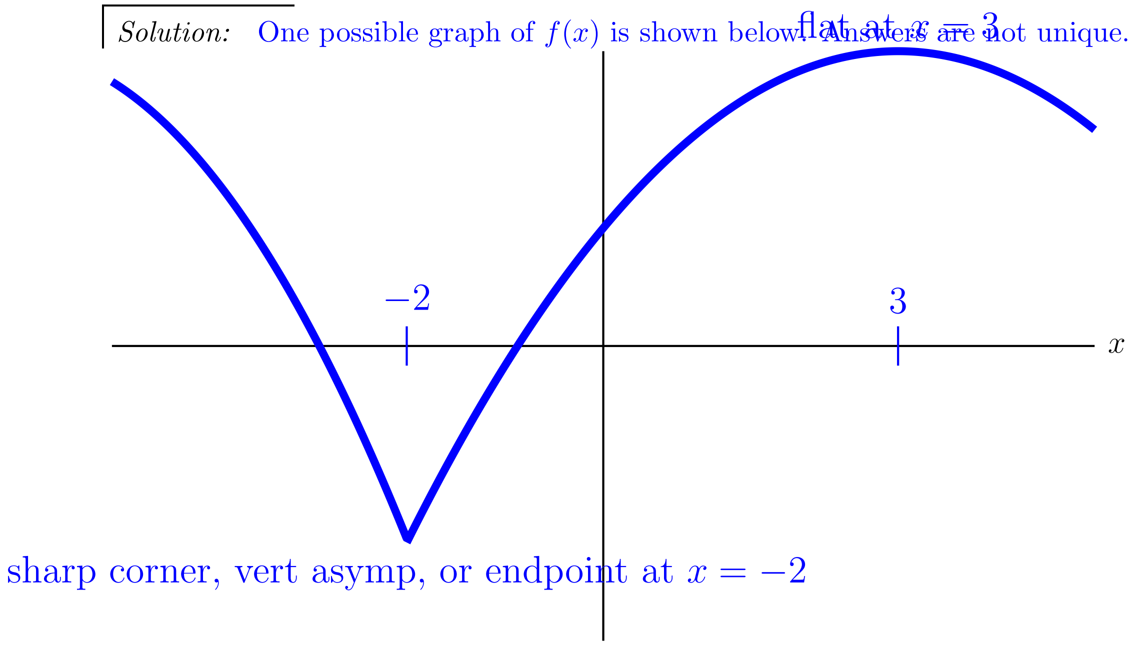
INFLECTION POINTS AT $(-\frac{1}{6}, e^{-\frac{1}{2}+B})$ and $(\frac{1}{6}, e^{-\frac{1}{2}+B})$.

4. [12 points]
The two parts below are independent. Be sure to label any relevant features of your graphs.

a. [6 points] Draw an example of a continuous function $f(x)$ such that

- f has a critical point at $x = -2$ and $f'(-2) \neq 0$, and
- f has a critical point at $x = 3$ and $f'(3) = 0$.

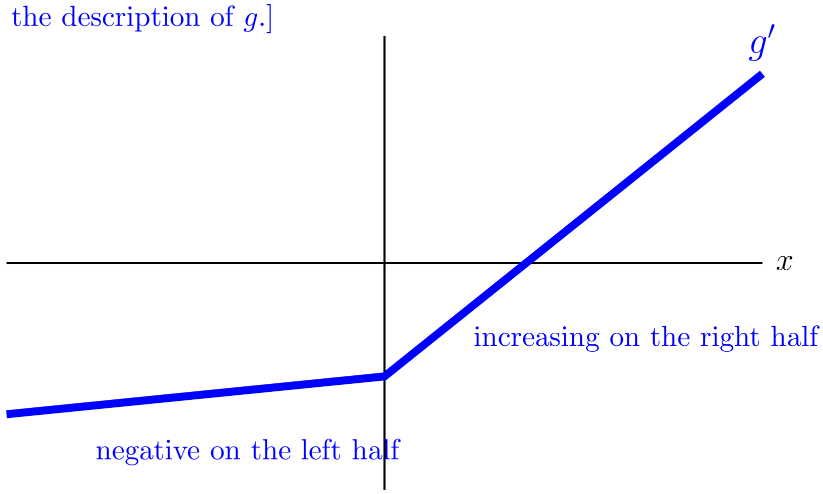
Solution: One possible graph of $f(x)$ is shown below. Answers are not unique.



b. [6 points] Draw the *derivative* of a function $g(x)$ satisfying

- g is decreasing on the interval $(-\infty, 0)$, and
- $g''(x) > 0$ when $x > 0$.

Solution: One possible graph of $g'(x)$ is shown below. Answers are not unique. [Note that in the graph below, $g(x)$ is decreasing beyond $x = 0$, but that does not contradict the description of g.]



5. [13 points] The equation below implicitly defines a hyperbola.

$$x^2 - y^2 = 2x + xy + y + 2.$$

a. [5 points] Find $\frac{dy}{dx}$.

Solution: We use implicit differentiation:

$$2x - 2y\frac{dy}{dx} = 2 + \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + \frac{dy}{dx} + 0.$$

Then solve for $\frac{dy}{dx}$:

$$\begin{aligned} 2x - 2 - y &= \frac{dy}{dx}(x + 2y + 1) \\ \frac{dy}{dx} &= \frac{2x - 2 - y}{x + 2y + 1} \end{aligned}$$

b. [4 points] Consider the two points $(4, 2)$ and $(2, -1)$. Show that one of these points lies on the hyperbola defined above, and one does not.

Solution: For the point $(4, 2)$, $x^2 - y^2 = 4^2 - 2^2 = 12$ and $2x + xy + y + 2 = 2(4) + 4(2) + 2 + 2 = 20$ are not equal, so $(4, 2)$ IS NOT on the hyperbola.

For the point $(2, -1)$, $x^2 - y^2 = 2^2 - (-1)^2 = 3$ and $2x + xy + y + 2 = 2(2) + 2(-1) - 1 + 2 = 3$ are equal, so $(2, -1)$ IS on the hyperbola.

c. [4 points] For the point in part (b) which lies on the hyperbola, find the equation of the line which is tangent to the hyperbola at this point.

Solution: From part (a),

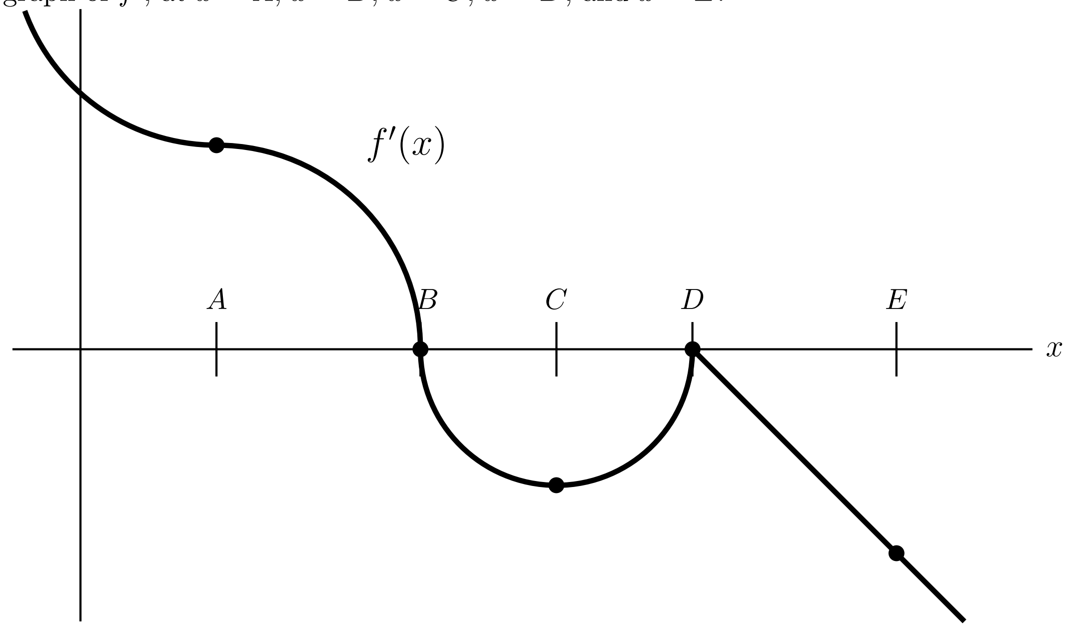
$$\frac{dy}{dx} = \frac{2x - 2 - y}{x + 2y + 1},$$

so

$$\frac{dy}{dx}\Big|_{(x,y)=(2,-1)} = \frac{2(2) - 2 - (-1)}{2 + 2(-1) + 1} = \frac{3}{1} = 3.$$

Then the equation of the tangent line is $y = 3(x - 2) - 1$ or $y = 3x - 7$.

6. [12 points] The *derivative* of a function f is graphed below. Five points are marked on the graph of f', at $x = A$, $x = B$, $x = C$, $x = D$, and $x = E$.



For each of the following, circle ALL answers which are correct. Each part has at least one correct answer. Pay careful attention to whether each question is asking about f, f', or f''.

a. [2 points] The function f' has a local minimum when ______________.

$x = A$ $x = B$ $\boxed{x = C}$ $x = D$ $x = E$

b. [2 points] The function f is increasing when ______________.

$\boxed{x = A}$ $x = B$ $x = C$ $x = D$ $x = E$

c. [2 points] The function f has a critical point when ______________.

$x = A$ $\boxed{x = B}$ $x = C$ $\boxed{x = D}$ $x = E$

d. [2 points] The global maximum of f on the interval $A \le x \le E$ occurs when ______________.

$x = A$ $\boxed{x = B}$ $x = C$ $x = D$ $x = E$

e. [2 points] The function f has an inflection point when ______________.

$x = A$ $x = B$ $\boxed{x = C}$ $\boxed{x = D}$ $x = E$

f. [2 points] The function f'' is undefined when ______________.

$x = A$ $\boxed{x = B}$ $x = C$ $\boxed{x = D}$ $x = E$

7. [16 points] Janet is an artist who produces and sells prints of her artwork. If Janet sells her prints for \$17 each, then she will sell 340 prints. Janet is considering whether she should change the price. She takes a survey and concludes that for each price increase of 75 cents, she will sell 10 fewer prints.

a. [4 points] Find a formula for Janet's revenue, $R(x)$, in terms of x, the number of 75 cent price increases.

Solution: Since revenue is price times quantity, we have

$$R(x) = (17 + 0.75x)(340 - 10x).$$

b. [4 points] Janet plans to produce exactly the number of prints that her survey predicts she will sell. Her costs include \$2 per print, along with \$500 in fixed costs. Find a formula for $C(x)$, Janet's total costs, in terms of x, the number of 75 cent price increases.

Solution: Fixed costs = \$500, cost per print = \$2 per print, and number of prints = $340 - 10x$, so

$$C(x) = 500 + 2(340 - 10x)$$

c. [8 points] Use the methods of calculus to determine what price Janet should set for her prints if she wants to maximize her profit.

Solution: Let $\pi(x) = \text{profit} = R(x) - C(x)$:

$$\pi(x) = (340 - 10x)(17 + 0.75x) - [500 + 2(340 - 10x)]$$

So,

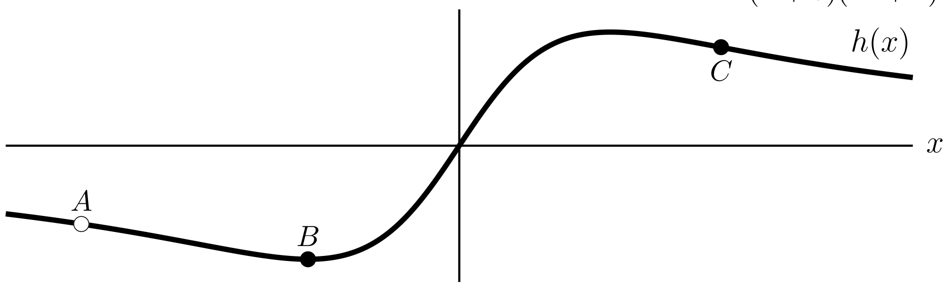
$$\begin{aligned} \pi'(x) &= (-10)(17 + 0.75x) + (340 - 10x)(0.75) - 2(-10) \\ &= -170 - 7.5x + 255 - 7.5x + 20 \\ &= -15x + 105. \end{aligned}$$

Note that $\pi'(x)$ is defined for all x, and $\pi'(x) = 0$ if $15x = 105$ or $x = 7$. Thus, we have one critical point at $x = 7$.

We must test the critical point. Using the second derivative test, we have $\pi''(x) = -15$ which is negative for all x, so $x = 7$ gives a local maximum. Since $\pi(x)$ is continuous, and we have only one critical point, $x = 7$ is a global max. Thus, $x = 7$ maximizes the profit.

The price that Janet should charge is $17 + 0.75(7) = 17 + 5.25 = \$22.25$.

8. [13 points] Below, there is a graph of the function $h(x) = \dfrac{2x^2 + 10x}{(x+5)(x^2+4)}$.



a. [3 points] The point A is a hole in the graph of h. Find the x- and y-coordinates of A.

Solution: Simplifying $h(x)$, we have $h(x) = \dfrac{2x(x+5)}{(x+5)(x^2+4)}$. Since the factor $(x+5)$ cancels, the hole occurs when $x = -5$. We look at the limit as x approaches -5 on the cancelled form to get the y-coordinate:

$$\lim_{x \to -5} h(x) = \lim_{x \to -5} \frac{2x}{x^2+4} = \frac{-10}{29},$$

Thus, $A = (-5, \frac{-10}{29})$.

b. [5 points] The point B is a local minimum of h. Find the x- and y-coordinates of B.

Solution: Using the quotient rule on the simplified form of h, we have $h'(x) = \dfrac{4 - x^2}{(x^2+4)^2}$. This is never undefined, and it is equal to zero when $4 - x^2 = 0$ or $x = \pm 2$. From the graph, we can see that the local minimum occurs at $x = -2$. The y-coordinate here is $y = \frac{-4}{8} = -\frac{1}{2}$, so $B = (-2, -\frac{1}{2})$.

c. [5 points] The point C is an inflection point of h. Find the x- and y-coordinates of C.

Solution: We use the quotient rule again to find $h''(x) = \dfrac{2x^3 - 24x}{(x^2+4)^3} = \dfrac{2x(x^2-12)}{(x^2+4)^3}$. This is never undefined, and it is zero when $2x(x^2 - 12) = 0$, i.e. when $x = 0, \pm 2\sqrt{3}$. From the graph, we see that our x-coordinate must be $+2\sqrt{3}$, and then $y = \frac{4\sqrt{3}}{16} = \frac{\sqrt{3}}{4}$, so $C = (2\sqrt{3}, \frac{\sqrt{3}}{4})$.