Math 115 — Final Exam

April 23, 2010

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. **Use the techniques of calculus to solve the problems on this exam.**

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	8	
6	14	
7	10	
8	12	
9	8	
Total	100	

1. [12 points]
For the following statements, select True if the statement is *ALWAYS* true, and select False otherwise. No explanations are required.

a. [2 points]
If f is a differentiable function and $\frac{f(5.1)-f(5)}{0.1} = -3$, then $f'(5) = -3$.

True $\boxed{\text{False}}$

b. [2 points]
If g is a continuous function, then

$$\int_1^{20} g(x)dx = \int_1^{-100} g(x)dx + \int_{-100}^{20} g(x)dx.$$

$\boxed{\text{True}}$ False

c. [2 points]
If h is an odd function and is continuous everywhere, then h is invertible.

True $\boxed{\text{False}}$

d. [2 points]
If k is a differentiable function and is always concave up,

then $k'(a) \leq \dfrac{k(b)-k(a)}{b-a}$ whenever $a < b$.

$\boxed{\text{True}}$ False

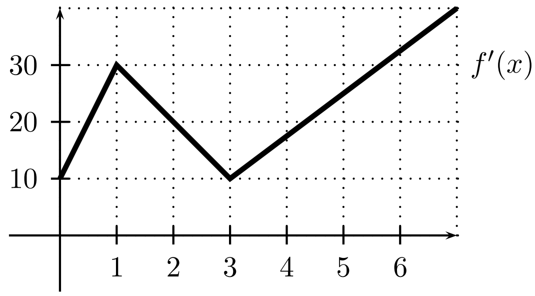
e. [2 points]
If ℓ is a continuous function, then
$\displaystyle\int_2^3 \ell(t)dt \leq \int_2^4 \ell(t)dt.$

True $\boxed{\text{False}}$

f. [2 points]
Suppose m is a twice differentiable function. If $m''(5) = 0$, then m does not have an inflection point at $x = 5$.

True $\boxed{\text{False}}$

2. [12 points]
Use the graph of the function f' and the table of values for the function g to answer the questions below. Each problem requires only a small amount of work, but you must show it.



x	-20	-10	0	10	20	30
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79

a. [3 points] Write a formula for the local linearization of g near $x = 10$ and use it to approximate $g(10.1)$.

Solution:

$$\begin{aligned} g(x) &\approx g(10) + g'(10)(x-10) \\ g(10.1) &\approx g(10) + g'(10)(10.1-10) \\ &= -18 + (-27)(0.1) = -20.7 \end{aligned}$$

b. [3 points] Using the table, estimate $g''(-10)$.

Solution: Note that $g''(x) \approx \dfrac{\Delta g'(x)}{\Delta x}$. We will use the two x-values closest to -10: $x = -20$ and $x = 0$.

$$g''(10) \approx \frac{g'(0) - g'(-20)}{0-(-20)} = \frac{-10-6}{0-(-20)} = \frac{-16}{20} = \frac{-4}{5} = -0.8$$

c. [3 points] If $f(3) = 30$, find the exact value of $f(1)$.

Solution: By the Fundamental Theorem of Calculus,

$$f(3) - f(1) = \int_1^3 f'(x)dx.$$

This integral represents the area under $f'(x)$ and above the x-axis between $x = 1$ and $x = 3$. Using basic geometry, this area is 40. Thus, $f(3) - f(1) = 40$, so $f(1) = f(3) - 40 = 30 - 40 = -10$.

d. [3 points] Given that $f(3) = 30$, find the exact value of $\int_1^3 g'(f(z))f'(z)dz$.
(Hint: use part (c).)

Solution: We recognize $g'(f(z))f'(z)$ as the derivative of $g(f(z))$, because of the chain rule. By the Fundamental Theorem, the integral above is equal to $g(f(3)) - g(f(1))$, which is equal to $g(30) - g(-10)$, using part c. Filling in values of g from the table, this is $-120 - 4 = -124$.

3. [12 points]
Scott is having a graduation party, and his mom wants to order individual cakes for the guests. Each cake is a right circular cylinder with radius R centimeters, height H centimeters, and volume 250 cubic centimeters. In addition,

- there is a fixed cost of \$3 per cake;
- the entire side of the cake will have maize icing with blue candy "M"s, which costs \$0.02 per square centimeter; and
- the entire top of the cake will have blue icing, which costs \$0.01 per square centimeter.

Recall that the volume of a right circular cylinder with radius R and height H is $V = \pi R^2 H$.

a. [4 points] Find a formula for the cost C of one cake, in terms of its radius R.

Solution:
$$\begin{aligned} C(R) &= 3 + 0.02(\text{area of the side}) + 0.01(\text{area of the top}) \\ &= 3 + 0.02(2\pi RH) + 0.01(\pi R^2) \end{aligned}$$

Solving for H in terms of R in the volume formula, gives $H = 250/\pi R^2$, so

$$\begin{aligned} C(R) &= 3 + 0.02(2\pi R\frac{250}{\pi R^2}) + 0.01\pi R^2 \\ &= = 3 + \frac{10}{R} + 0.01\pi R^2. \end{aligned}$$

b. [8 points] What radius and height should Scott's mom choose for the cakes if she wishes to minimize her costs? What is the minimum price for one cake? (To get credit, you must fully justify your answer using algebraic work.)

Solution: We need to find critical points to find the minimum cost, using the first derivative:

$$C'(R) = \frac{-10}{R^2} + 0.02\pi R.$$

A reasonable domain is $R > 0$, and C' is only undefined when $R = 0$, which is outside the domain. We set $C' = 0$ to find any other critical points.

$$\begin{aligned} \frac{-10}{R^2} + 0.02\pi R &= 0 \\ -10 + 0.02\pi R^3 &= 0 \\ R^3 &= \frac{10}{0.02\pi} \\ R &= \left(\frac{10}{0.02\pi}\right)^{1/3} \approx 5.4193 \end{aligned}$$

Since $C''(R) = \frac{20}{R^3} + 0.02\pi$, we have $C''(5.4193) > 0$, so $R = 5.4193$ is a local minimum. It must also be a global minimum, since C is continuous on the domain and there is only one critical point. Then $H = \frac{250}{\pi 5.4193^2} \approx 2.7096$, and $C = 3 + \frac{10}{5.4193} + 0.01\pi(5.4193)^2 \approx 5.77$.

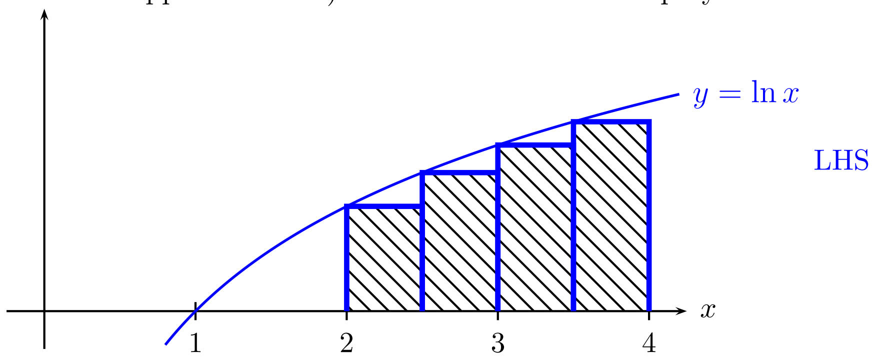
radius = 5.4193*cm* *height* = 2.7096*cm* *cost* = \$5.77

4. [12 points]

a. [6 points] Using 4 equal subdivisions, find a Riemann sum which is an underestimate for

$$\int_2^4 \ln(x)dx.$$

Sketch a graphical representation of your Riemann sum on the axes below, and write "LHS" or "RHS" next to your figure to indicate whether you are using a left-hand sum or a right-hand sum. Write out the terms of the Riemann sum using exact values (no calculator approximations). There is no need to simplify the sum.



$\int_2^4 \ln(x)dx \approx \underline{0.5\ln 2 + 0.5\ln 2.5 + 0.5\ln 3 + 0.5\ln 3.5}$

b. [3 points]

Show that $\int \ln(x)dx = x\ln(x) - x + C$, where C is a constant.

Solution: We need to check that $\frac{d}{dx}(x\ln(x) - x + C) = \ln(x)$. Using the product rule, we have

$$\frac{d}{dx}(x\ln(x) - x + C) = \left(x \cdot \frac{1}{x} + 1 \cdot \ln x\right) - 1 + 0 = 1 + \ln(x) - 1 = \ln(x).$$

c. [3 points]

Using part (b), find the exact value of the integral $\int_2^4 \ln(x)dx$.

Solution: By the Fundamental Theorem and part (b),

$$\begin{aligned}\int_2^4 \ln(x)dx &= [4\ln(4) - 4 + C] - [2\ln(2) - 2 + C] \\ &= 4\ln 4 - 2\ln 2 - 2 \\ &= \ln 4^3 - 2.\end{aligned}$$

5. [8 points]
Suppose that the *derivative* of a continuous function H is given by the formula

$$H'(t) = \frac{e^{t(t+1)}(t-3)(t+450)}{(4t-100)^{\frac{2}{3}}}.$$

Find all values of t which are critical points of the original function H. Use the first derivative test (and explain your work) to identify each critical point as a local maximum, local minimum, or neither.

Solution: The critical points of H are the values of t for which H' is zero or undefined. Now, $H'(t) = 0$ when $t = 3$ or $t = -450$, and $H'(t)$ is undefined for $t = 25$. Now we need to check whether each of these is a local max, local min, or neither.
These three critical points split the domain into four intervals: $(\infty, -450)$, $(-450, 3)$, $(3, 25)$, and $(25, \infty)$. We will find out the sign of H' for each interval by checking the sign of each factor.

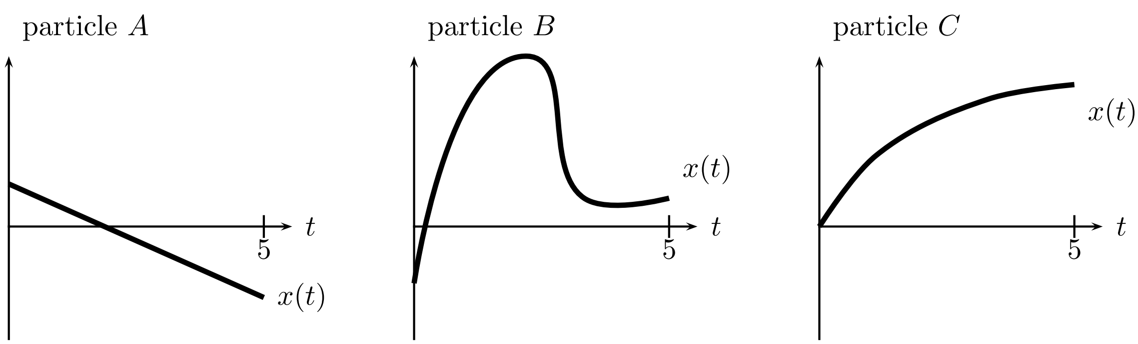
interval	$H'(t)$	H is...
$(\infty, -450)$	$\frac{+--}{+} = +$	increasing
$(-450, 3)$	$\frac{+-+}{+} = -$	decreasing
$(3, 25)$	$\frac{+++}{+} = +$	increasing
$(25, \infty)$	$\frac{+++}{+} = +$	increasing

Since H is increasing just before $t = -450$ and decreasing right after, $t = 450$ is a local maximum.

Since H is decreasing just before $t = 3$ and increasing right after, $t = 3$ is a local minimum.

Since H is increasing on both sides of $t = 25$, we know $t = 25$ is neither a local max nor a local min.

6. [14 points]
Each graph below shows the *position* of a particle moving along the x-axis as a function of time, for $0 \leq t \leq 5$. The vertical scales are all the same. Using the position graphs, answer the questions below. No work is required.



Which one of the three particles (A, B, C)...

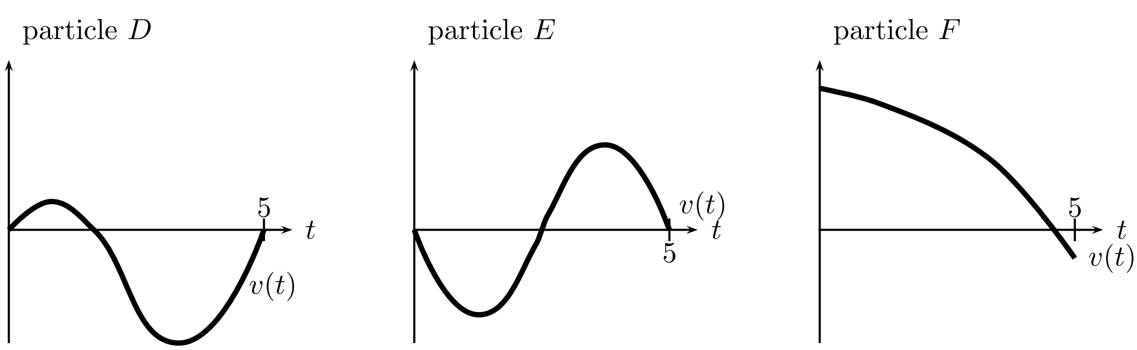
a. [2 points] has the greatest initial velocity? ____B____

b. [2 points] has zero acceleration? ____A____

c. [2 points] has the greatest average velocity? ____C____

d. [2 points] travels the greatest distance? ____B____

Each graph below shows the *velocity* of a particle moving along the x-axis as a function of time, for $0 \leq t \leq 5$. Positive velocity indicates that the particle is traveling to the right. Negative velocity indicates travel to the left. The vertical scales are all the same. These are not the same particles as above. Using the velocity graphs, answer the questions below. No work is required.



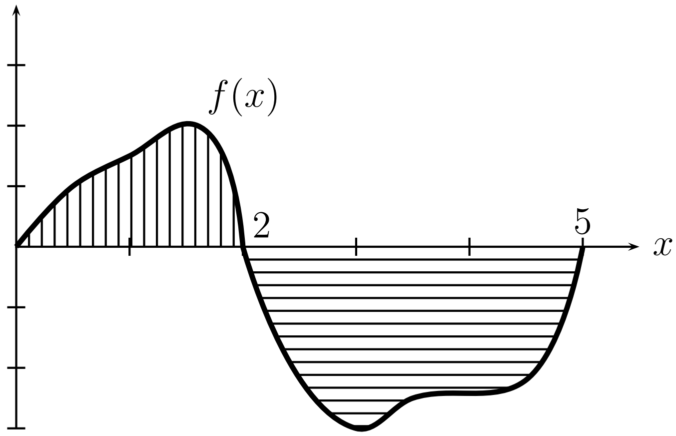
Which one of the three particles (D, E, F)...

e. [2 points] returns to its starting position when $t = 5$? ____E____

f. [2 points] has the greatest average velocity? ____F____

g. [2 points] ends up farthest to the left of where it started? ____D____

7. [10 points] Suppose that f is an even function. A portion of f is graphed below.



The area of the shaded region between $x = 0$ and $x = 2$ (with vertical stripes) is 3 units, and the area of the shaded region between $x = 2$ and $x = 5$ (with horizontal stripes) is 8 units. Find exact values for each of the following integrals. If it is not possible to find the exact value, write "insufficient information".

a. [2 points] $\int_{-2}^{2} f(x)dx$

Solution: Since f is even, $\int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)dx = 2(3) = 6$.

b. [2 points] $\int_{0}^{5} |f(x)|dx$

Solution: Since we are integrating the absolute value of f, we want the total area between f and the x-axis, between $x = 0$ and $x = 5$, which is $3 + 8 = 11$.

c. [2 points] $\int_{0}^{1} f(2t)dt$

Solution: Since $f(2t)$ is only half as wide as $f(t)$, the shaded area on the left gets compressed to half its width and thus half its area. Thus, $\int_{0}^{1} f(2t)dt = \frac{1}{2}(3) = 1.5$.

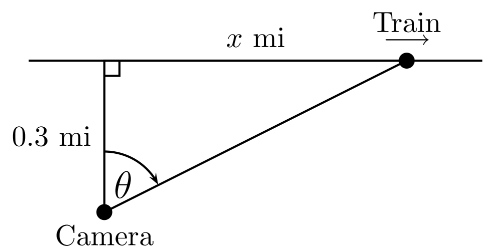
d. [2 points] $\int_{5}^{8} f(t-3)dt$

Solution: The function $f(t-3)$ is simply $f(t)$ shifted 3 units to the right. Thus, $\int_{5}^{8} f(t-3)dt = \int_{2}^{5} f(t)dt = -8$.

e. [2 points] $\int_{5}^{2} 9f(z)dz$

Solution: The function $9f(z)$ is 9 times as tall as $f(z)$, so $\int_{5}^{2} 9f(z)dz = 9\int_{5}^{2} f(z)dz = 9 \cdot 8 = 72$.

8. [12 points]
A train is traveling eastward at a speed of 0.4 miles per minute along a long straight track, and a video camera is stationed 0.3 miles from the track, as shown in the figure. The camera stays in place, but it rotates to focus on the train as it moves.
Suppose that t is the number of minutes that have passed since the train was directly north of the camera; after t minutes, the train has moved x miles to the east, and the camera has rotated θ radians from its original position.



a. [3 points] Write an equation that expresses the relationship between x and θ.

Solution:

$$\tan(\theta) = \frac{x}{0.3}, \text{ or } \theta = \arctan\left(\frac{x}{0.3}\right)$$

b. [4 points] Suppose that seven minutes have passed since the train was directly north of the camera. How far has the train moved in this time, and how much has the camera rotated?

Solution: The train's velocity is constant (i.e. $\frac{dx}{dt} = 0.4$), so we can use the formula *distance* $=$ *velocity* $\cdot$ *time*. Thus, the train has moved $\left(0.4\frac{\text{mi}}{\text{min}}\right)(7\text{min}) = 2.8$ miles.

Using the fact that $x = 2.8$, we have $\theta = \tan^{-1}(\frac{2.8}{0.3}) \approx 1.4641$, so the camera has rotated 1.4641 radians in the clockwise direction.

c. [5 points] How fast is the camera rotating (in radians per minute) when $t = 7$?

Solution: We want to find $\frac{d\theta}{dt}$. To do so, we will (implicitly) take the derivative of our equation from part (a), with respect to t.

$$\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\left(\frac{x}{0.3}\right), \text{ or } \frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = \frac{1}{0.3}\frac{dx}{dt}$$

We can then plug in $\frac{dx}{dt} = 0.4$ and $\theta = 1.4641$:

$$\frac{1}{\cos^2(1.4641)}\frac{d\theta}{dt} = \frac{0.4}{0.3}, \text{ so } \frac{d\theta}{dt} = \frac{0.4}{0.3}\cos^2(1.4641) \approx 0.01513.$$

Thus, the camera is rotating at a speed of 0.01513 radians per minute (in the clockwise direction) when $t = 7$.

9. [8 points]
Suppose that the standard price of a round-trip plane ticket from Detroit to Paris, purchased t days after April 30, is $P(t)$ dollars. Assume that P is an invertible function (even though this is not always the case in real life).

In the context of this problem, give a practical interpretation for each of the following:

a. [2 points] $P'(2) = 55$

Solution: The standard price of a round-trip ticket from Detroit to Paris is approximately $55 more if the ticket is purchased on May 3 than if it is purchased on May 2.

b. [2 points] $P^{-1}(690)$

Solution: The standard price of a round-trip ticket from Detroit to Paris is $690 if it is purchased $P^{-1}(690)$ days after April 30.

c. [2 points] $\int_5^{10} P'(t)dt$

Solution: The standard price of a round-trip ticket from Detroit to Paris changes by $\int_5^{10} P'(t)dt$ dollars between May 5 and May 10. (If the integral is positive, it will be a price increase. If the integral is negative, it will be a price decrease.)

d. [2 points] $\frac{1}{5}\int_5^{10} P(t)dt$

Solution: This is the average standard price (in dollars) of a round-trip ticket from Detroit to Paris purchased between May 5 and May 10.