Math 115 — Final Exam

April 25, 2011

Name: ______________________________

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. Note that problems 6–9 will be graded giving very little partial credit.

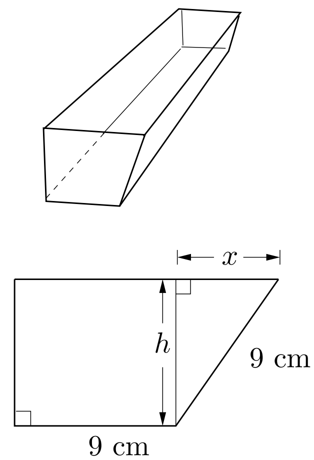
Problem	Points	Score
1	10	
2	13	
3	10	
4	12	
5	11	
6	10	
7	10	
8	14	
9	10	
Total	100	

1. [10 points] Find a formula for a function of the form

$$f(x) = \frac{1}{a + x + bx^2}$$

which has a local minimum at $(2, 1/2)$. Be sure to show that your function has a minimum at $(2, 1/2)$.

2. [13 points] A rain gutter attaches to the edge of a roof and collects the rain that falls on the roof. A common gutter design is shown in the figure to the right, and has a trapezoidal cross-section (also shown). In this problem we consider a gutter with base and side length 9 cm, as shown.



a. [1 point] Write an equation which relates the length x to the height h.

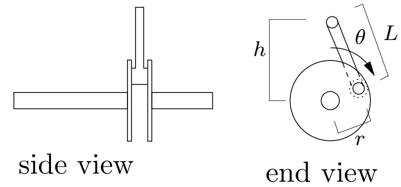
b. [4 points] Using your equation from (a), write an equation for the cross-sectional area of the gutter as a function of the length x (note that the area is the sum of a rectangular and right-triangular region).

c. [8 points] Find the length x that gives the maximum cross-sectional area. Be sure to show work that demonstrates that you have found the maximum.

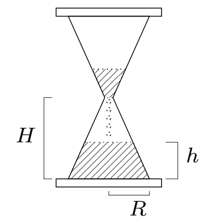
3. [10 points] For each of the following determine the indicated quantity.

a. [4 points] In an internal combustion engine, pistons are pushed up and down by a crank shaft similar to the diagram shown to the right. As the shaft rotates the height of the piston, h, is related to the rotational angle θ of the shaft by $h = r\cos\theta + \sqrt{L^2 - r^2\sin^2\theta}$, where r and L are constant lengths. If $r = 10$ cm, $L = 15$ cm, and h is decreasing at a rate of 5000 cm/s when $\theta = 3\pi/4$, how fast is θ changing then?

Crank shaft diagram (part a)



Hourglass diagram (part b)



b. [6 points] The lower chamber of an hourglass is shaped like a cone with height H in and base radius R in, as shown in the figure to the right, above. Sand falls into this cone. Write an expression for the volume of the sand in the lower chamber when the height of the sand there is h in *(Hint: A cone with base radius r and height y has volume $V = \frac{1}{3}\pi r^2 y$, and it may be helpful to think of a difference between two conical volumes.)*. Then, if $R = 0.9$ in, $H = 2.7$ in, and sand is falling into the lower chamber at $2\,\text{in}^3/\text{min}$, how fast is the height of the sand in the lower chamber changing when $h = 1$ in?

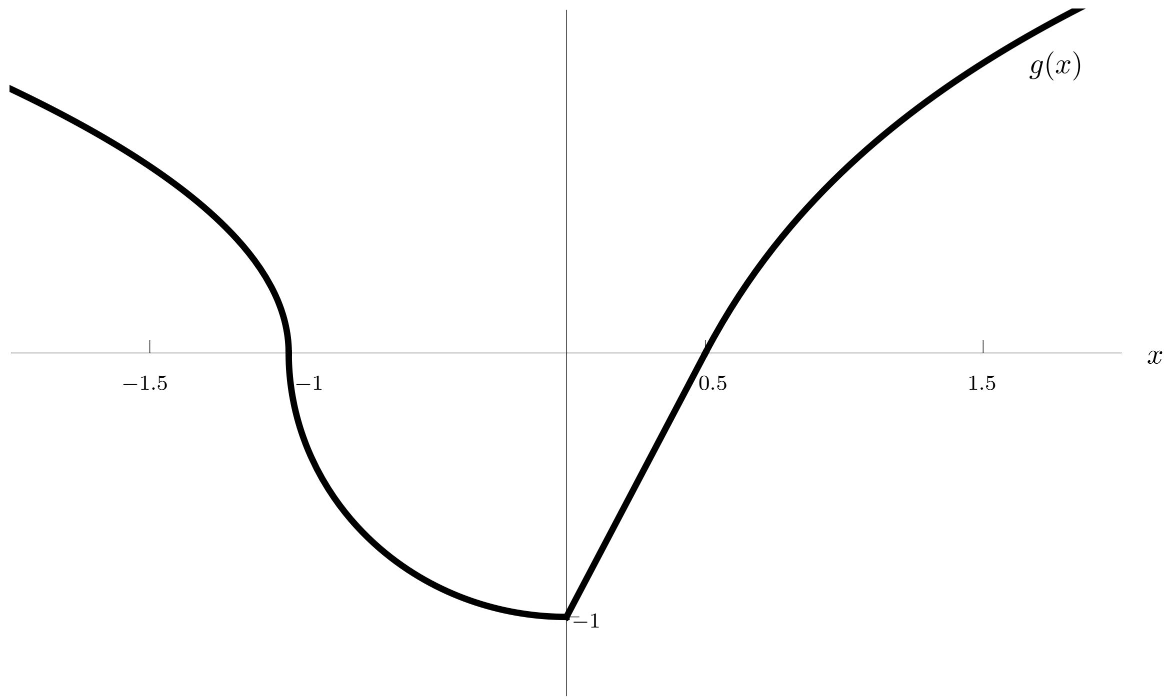
4. [12 points] Suppose $P(\theta)$ is the power, in kilojoules per hour (kJ/h), produced by a solar panel when the angle between the sun and the panel is θ, measured in degrees. Suppose $C(t)$ is the power, in kJ/h, produced by the solar panel t hours after sunrise on a typical summer day. Give practical interpretations of the following.

a. [4 points] $P'(30) = 9$.

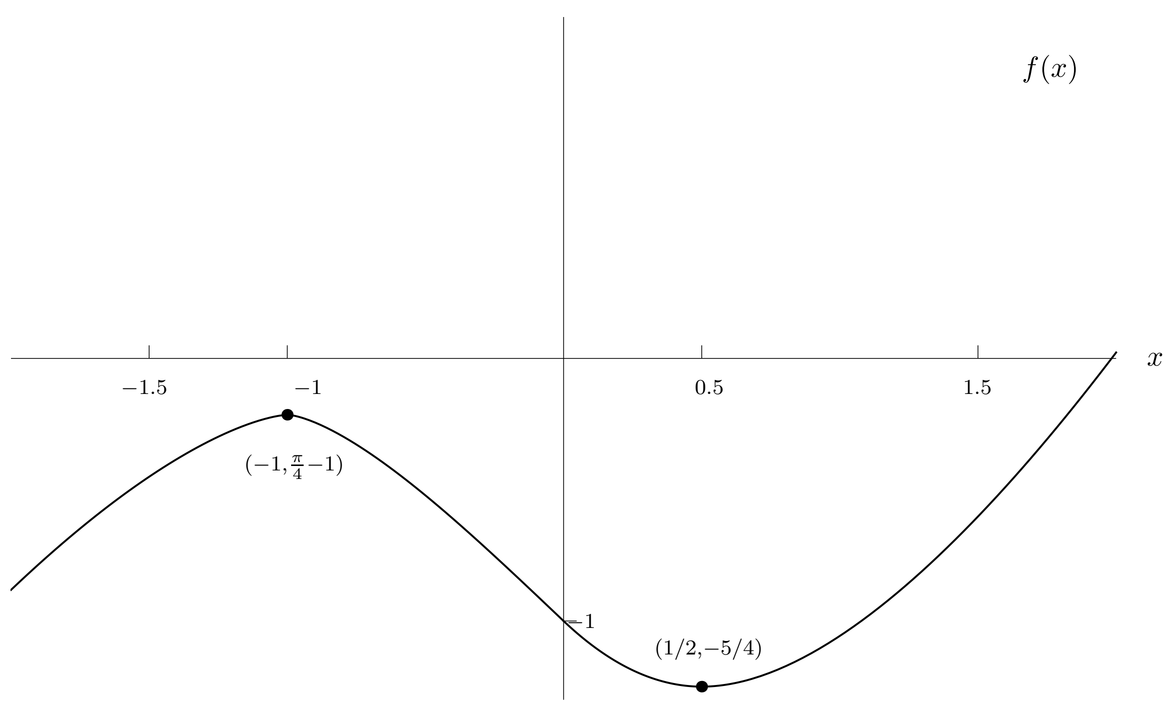
b. [4 points] $\int_0^2 C(t)\,dt = 270$.

c. [4 points] $\frac{1}{12}\int_0^{12} C(k)\,dk = 288$.

5. [11 points] A function $g(x)$ is graphed below. The curve forms a quarter circle between $x = -1$ and $x = 0$ and a line between $x = 0$ and $x = 0.5$



On the axes below, sketch a well-labeled graph of $f(x)$, an antiderivative of $g(x)$, satisfying $f(0) = -1$. Be sure to label the coordinates of $f(x)$ at $x = -1$ and $x = 0.5$.



6. [10 points] The table below gives the expected growth rate, $g(t)$, in ounces per week, of the weight of a baby in its first 54 weeks of life (which is slightly more than a year).[1] Assume for this problem that $g(t)$ is a decreasing function.

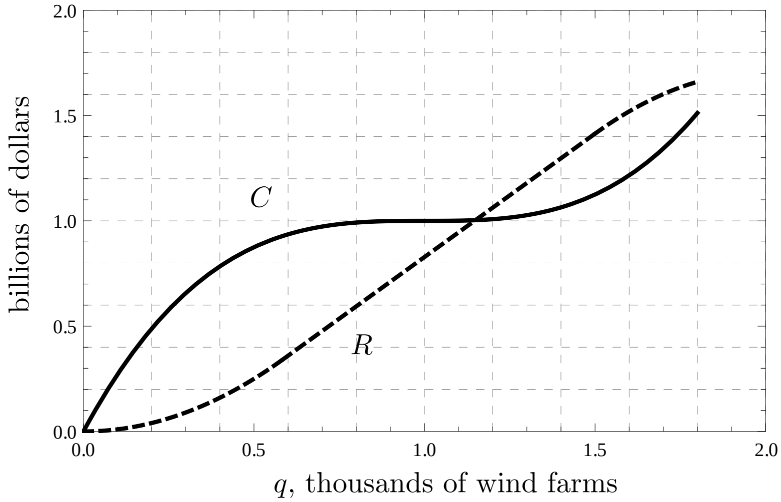
week t	0	9	18	27	36	45	54
growth rate $g(t)$	6	6	4.5	3	3	3	2

a. [6 points] Using six subdivisions, find an overestimate and underestimate for the total weight gained by a baby over its first 54 weeks of life.

b. [4 points] How frequently over the 54 week period would you need the data for $g(t)$ to be measured to find overestimates and underestimates for the total weight gain over this time period that differ by 0.5 lb (8 oz)?

[1]Riordan J. *Breastfeeding and Human Lactation*, 3rd ed. Boston: Jones and Bartlett, 2005, p.103, 512-513.

7. [10 points] Airwatt Construction company builds large-scale farms of wind turbines. A graph of cost C and revenue R for the company at different production levels q is shown below. Here cost and revenue are measured in billions of dollars and production level is measured in thousands of wind farms. *In each of the following parts, be sure it is clear how you obtain your answers.*



a. [3 points] Approximate the marginal revenue at $q = 0.8$. Show how you obtain your estimate.

b. [3 points] Approximate the cost of producing one additional wind farm when $q = 1.6$.

c. [4 points] Approximate the maximum profit which can be achieved by Airwatt. At what production level does this occur?

8. [14 points] A car speeds up at a constant rate from 10 to 70 mph over a period of half an hour, between $t = 0$ and $t = 1/2$. Its fuel efficiency, $E(v)$, measured in miles per gallon, depends on its speed, v, measured in miles per hour.

a. [4 points] Write an integral which gives the total distance traveled by the car during the half hour.

b. [5 points] Write an integral which gives the average fuel efficiency of the car during the half hour.

c. [5 points] For speeds v greater than 70 mph suppose the relationship between E and v is given by

$$E(v) = 2 + v^{-av}$$

for some constant a. Using this formula, write an expression for the definition of the derivative $E'(82)$. Do not evaluate your expression.

9. [10 points] For each of the statements below, circle TRUE if the statement is always true and circle FALSE otherwise. The letters a, b and c below represent real number constants. Any ambiguous marks will be marked as incorrect. No partial credit will be given on this problem.

a. [2 points] Let $f(x)$ and $g(x)$ be continuous functions which are defined for all real numbers. If $f(x) \geq g(x)$ for all real numbers x, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$ whenever $a < b$.

True False

b. [2 points] If a is a positive, then the function $h(x) = \frac{\ln(ax^2)+x}{x}$ is an antiderivative of $j(x) = \frac{2-\ln(ax^2)}{x^2}$.

True False

c. [2 points] Suppose a differentiable function $\ell(x)$ is concave down and defined for all real numbers. If $a < b$, then

$$\frac{\ell(b) - \ell(a)}{b - a} < \ell'(b).$$

True False

d. [2 points] If $x = a$ is a critical point of a function $m(x)$, then $m'(a) = 0$.

True False

e. [2 points] If $n(x)$ and $p(x)$ are continuous functions which are defined for all real numbers, then

$$\int_a^b (c\,n(x) - p(x))\,dx = c\int_a^b n(x)\,dx + \int_b^a p(x)\,dx$$

True False