Math 115 — First Midterm

February 7, 2012

Name: EXAM SOLUTIONS

Instructor: __________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	13	
3	13	
4	15	
5	12	
6	9	
7	9	
8	9	
9	10	
Total	100	

1. [10 points] Kimberly is walking from her home to the local juice bar, and the function $D(t)$ gives her distance in meters from the juice bar t minutes after the moment of her departure from home. She walks in a straight line towards her destination, never stopping or backtracking.

a. [2 points] What is the sign of $D'(1)$, assuming that Kimberly is still walking 1 minute after having left home?

Solution: Negative because the distance between Kimberly and the juice bar is decreasing.

b. [2 points] Let c be a positive constant less than the distance between Kimberly's home and the juice bar. What mathematical expression gives Kimberly's velocity at the moment when she is c meters from the juice bar?

Solution:

$$D'(D^{-1}(c)) \text{ meters/minute.}$$

$D'(t)$ gives Kimberly's velocity t minutes after she departs. $D^{-1}(c)$ gives the time when she is c meters from the juice bar.

c. [2 points] For the rest of this problem, assume that $D(t)$ is a linear function. Five minutes after leaving, Kimberly is 600 meters from the juice bar, and another three minutes after that, she is only 350 meters from the juice bar. Write an explicit formula for $D(t)$.

Solution: $D(t)$ is a line which goes through the points $(5, 600)$ and $(8, 350)$. The slope of the line is

$$\frac{350 - 600}{8 - 5} = -\frac{250}{3} \approx -83.33.$$

Using point slope form, the equation for this line is

$$D(t) = -\frac{250}{3}(t - 5) + 600.$$

d. [2 points] How long does it take Kimberly to get to the juice bar?

Solution: The answer here is the time when Kimberly's distance from the juice bar is zero. Setting

$$0 = -\frac{250}{3}(t - 5) + 600$$

we get

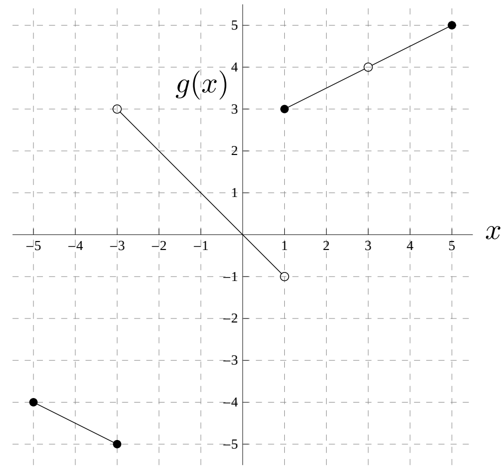
$$t = 12.2 \text{ minutes}$$

e. [2 points] How far away from the juice bar does Kimberly live?

Solution: The answer here is $D(0) = 1016.67$ meters .

2. [13 points] Below is a table of values for an invertible, differentiable function $f(x)$ and the graph of a function $g(x)$. Use these to answer the following questions:

x	0	1	2	3	4	5
$f(x)$	8	7	3	2	1.5	1



a. [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of g.

Solution: 3

b. [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of g^{-1}.

Solution: Anything in $(-4, -1] \cup \{4\}$. For example, 4.

c. [8 points] Evaluate the following:

(i) $f(f(5))$

Solution: $f(f(5)) = f(1) = 7$.

(ii) $g^{-1}(f^{-1}(2))$

Solution: $g^{-1}(f^{-1}(2)) = g^{-1}(3) = 1$.

(iii) $\lim_{x\to 3} g(x)$

Solution: 4, found from graph of g.

(iv) $g'(1 + f(2))$

Solution: $g'(1 + f(2)) = g'(4) = \dfrac{1}{2}$ looking at the slope on the graph of g at $x = 4$.

d. [3 points] Approximate $f'(3)$. (Be sure to show your work.)

Solution: Acceptable answers: -1, $-\frac{1}{2}$, $-\frac{3}{4}$. Found approximating the derivative via a difference quotient.

3. [13 points] A wedge of cheese in Zack's refrigerator has become home to a colony of bacteria. Let $A(t)$ be the surface area of the colony (in cm^2) t days after the expiration date of the cheese.

a. [4 points] For the first 20 days after the expiration date, the surface area of the colony grows exponentially. During this time, it takes the colony 5 days to double. Write a formula for $A(t)$ on the domain $0 \le t \le 20$. (Your formula may involve an unknown constant, but be sure to specify what this constant means in terms of bacteria.)

Solution:

$$A(t) = A_0 2^{t/5}, \text{ where } A_0 \text{ is the initial surface area of the colony.}$$

Beginning with $A(t) = A_0 b^t$ we have that the surface area of the bacteria doubles in 5 days, so we set $2A_0 = A_0 b^5$. Then $2^{1/5} = b$.

b. [3 points] How many days does it take for the surface area of the colony to triple? (Your answer does not need to be a whole number.)

Solution:

$$5\frac{\log(3)}{\log(2)} \approx 7.9248 \text{ days}$$

Beginning with our equation from (a) we set $3A_0 = A_0 2^{t/5}$. Taking ln of both sides and simplifying we have $\ln 3/\ln 2 = t/5$.

c. [3 points] Twenty days after the expiration date, the bacteria mysteriously begin to die off. The surface area of the colony on the cheese decreases linearly at a rate of $0.3 \text{ cm}^2/\text{day}$ starting at $t = 20$, and by $t = 22$ the surface area has fallen to 9 cm^2. Given that $A(t)$ is a continuous function, what was the surface area of the colony on the expiration date of the cheese?

Solution: Working backwards from $t = 22$ we have that the surface area was 0.6 cm^2 more at $t = 20$ than at $t = 22$. This means it was 9.6 cm^2 at $t = 20$ where the exponential growth stopped. Setting

$$9.6 = A_0(2)^{20/5}$$

we have

$$A_0 = 0.6\text{cm}^2$$

d. [3 points] What is $A'(20)$, or is it undefined? Justify your answer with a rough sketch of $A(t)$.

Solution:
It is undefined because $A(t)$ is not differentiable due to corner on its graph at $t = 20$.

4. [15 points] In the following problems, circle all of the statements that *could* be true and draw a line through all of the statements that *could not* be true, based on the given information. (Every statement should be either circled or crossed out.) No explanation is necessary.

a. [3 points] A brief table of values for $f(x)$ and $g(x)$ is given, rounded to 4 decimal places:

x	$f(x)$	$g(x)$
1.25	2.4414	1.1265
1.5	5.0625	1.1547
1.75	9.3789	1.1836

- $f(x)$ is exponential and $g(x)$ is a power function.
- $f(x)$ is a power function and $g(x)$ is exponential.
- $f(x)$ and $g(x)$ are both exponential.

Solution:

F, T, F

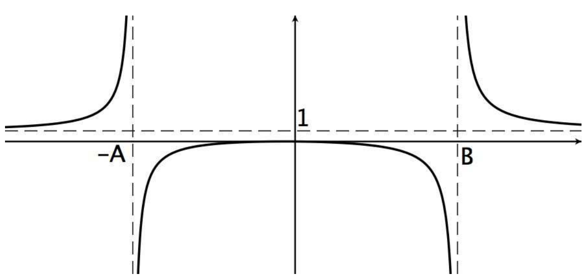
b. [4 points] Suppose that $f(x)$ is a continuous function and $\lim_{x\to\infty} f(x) = 2$.

- For all $x > 10$, $f''(x) > 0$ and $f'(x) > 0$.
- For all $x > 10$, $f''(x) > 0$ and $f'(x) < 0$.
- For all $x > 10$, $f''(x) < 0$ and $f'(x) > 0$.
- For all $x > 10$, $f''(x) < 0$ and $f'(x) < 0$.

Solution:

F, T, T, F

c. [4 points] A rational function $r(x)$ is graphed below (with $A \neq B$):

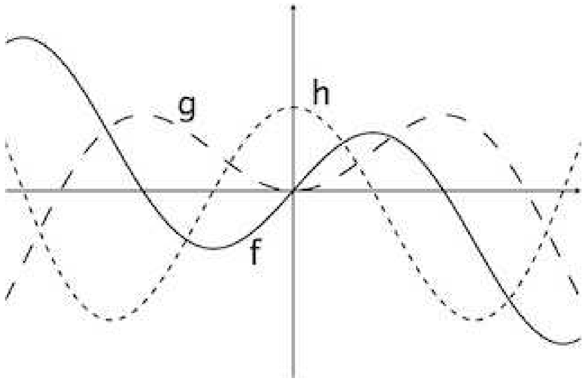


- $r(x) = \dfrac{x^2}{(x+A)(x-B)}$
- $r(x) = \dfrac{x}{(x+A)(x-B)}$
- $r(x) = \dfrac{4x^2}{(x+A)(x-B)}$
- $r(x) = \dfrac{x^2}{(x-A)(x+B)}$

Solution:

T,F,F,F

d. [4 points] Consider the functions $f(x), g(x)$, and $h(x)$ graphed below.



- $f(x) = h'(x)$
- $h(x) = f'(x)$
- $h(x) = g''(x)$
- $f(x) = g'(x)$

Solution:

F, T, T, T

5. [12 points] Preparing a pot of soup for dinner, Billy heats the soup to boiling and then removes it from the stove. The function $H(t)$ gives the temperature of the soup in °F as a function of the number of minutes since it was removed from the stove. Assume that $H(0) = 212$ and $H'(0) = -2.7$, and that Billy's kitchen is carefully air-conditioned to remain at a comfortable 68 °F at all times. Throughout this problem, be sure to **include units** in your answers, where applicable.

a. [3 points] Approximate $H(1.5)$.

Solution:

$$H(1.5) \approx 212 + 1.5(-2.7) = 207.95\ ^\circ\mathrm{F}$$

b. [3 points] Five minutes after removing the soup from the stove, Billy remarks to himself: "In the next 30 seconds, I expect the soup to cool by about 0.875 °F." Since he is both an excellent chef and a student of Math 115, his statement is consistent with the actual value of the derivative of the function H. Based on this information, find $H'(5)$, and justify your answer.

Solution: If in 30 seconds the soup will cool by 0.875 °F, in a minute we would expect it to cool by approximately 1.75 °F, so

$$H'(5) = -1.75\ ^\circ\mathrm{F/minute}$$

c. [3 points] Assume that the concavity of $H(t)$ is the same on its entire domain. Based on your answer to part (b) and the given information, do you expect that the function $H(t)$ is concave up or concave down? Briefly explain your answer.

Solution: Between $t = 0$ and $t = 5$ the derivative has increased from approximately -2.7 to approximately -1.75, so $H(t)$ should be concave up.

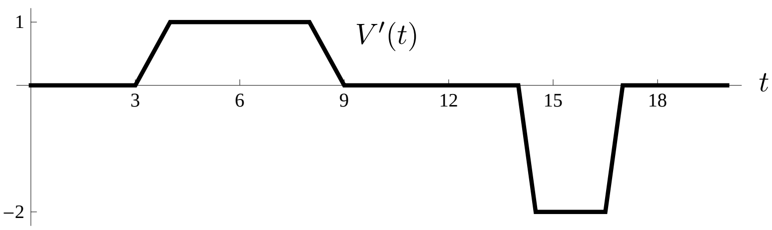
d. [3 points] Called off on important business, Billy leaves the pot of soup uneaten. Approximate $H'(300)$. (You may use the practical interpretation of $H(t)$, but be sure to explain your answer.)

Solution:

$$H'(200) \approx 0\ ^\circ\mathrm{F/minute}.$$

The function $H(t)$ is concave up and decreasing, so after a long period of time, the slope of the graph should approach zero.

6. [9 points] Trini is filling a bucket with water in order to water her garden. She brings the bucket to her garden hose, fills it up, walks to her row of tomato plants, and gradually empties it. If $V(t)$ is the volume of water in the bucket at time t, then the graph below shows the derivative, $V'(t)$:



In the following questions, **no explanation** is necessary.

a. [2 points] At what time does Trini begin to fill the bucket?

Solution:

$$t = 3$$

b. [2 points] At what time does Trini stop filling the bucket?

Solution:

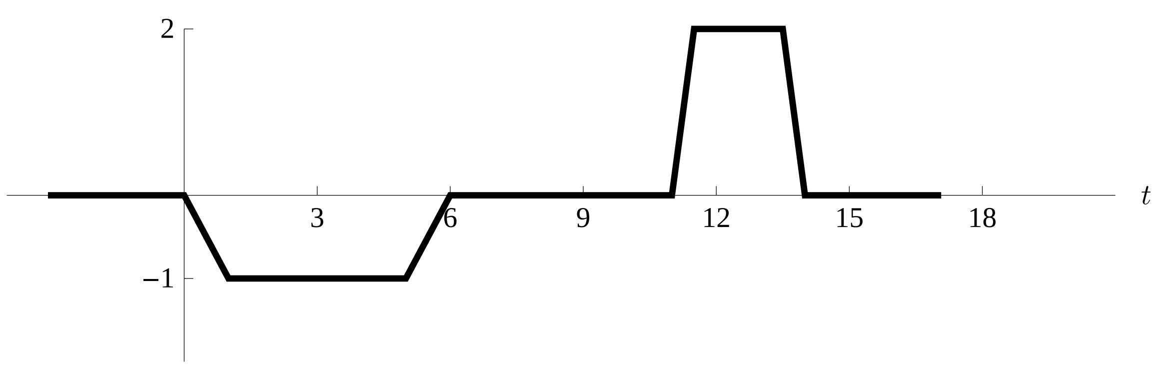
$$t = 9$$

c. [2 points] At what time does Trini start to empty the bucket?

Solution:

$$t = 14$$

d. [3 points] On the axes below, sketch a well-labeled graph of $-V'(t+3)$. (Remember that the figure above gives the graph of $V'(t)$.)



7. [9 points] Enjoying breakfast outdoors in a coastal Mediterranean town, Tommy notices a ship that is anchored offshore. The ship is stationed above a reef which lies below the surface of the water, and a series of waves causes its height to oscillate sinusoidally with a period of 6 seconds. When Tommy begins observing, the hull of the ship is at its highest point, 20 feet above the reef. After 1.5 seconds, the hull is 11 feet above the reef.

a. [6 points] Write a function $h(t)$ that gives the height of the ship's hull above the reef t seconds after Tommy begins observing.

Solution:

$$h(t) = 9\cos\left(\frac{\pi}{3}t\right) + 11.$$

The function starts out at its maximum, so we will use cosine with no horizontal shift making our formula $h(t) = A\cos(Bt) + C$. The period is given to be 6. This means $B = 2\pi/6 = \pi/3$. When $t = 1.5$, we have

$$11 = h(1.5) = A\cos\left(\frac{\pi}{3}\cdot 1.5\right) + C = C$$

and so when $t = 0$ we have

$$20 = h(0) = A + C = A + 11.$$

Solving this, we have $A = 9$.

b. [3 points] According to your function, will the hull of the ship hit the reef? Explain.

Solution: No. At it's lowest point at $t = 3$, $h(3) = 2$, so the ship remains two feet above the reef.

8. [9 points] A certain company's revenue R (in thousands of dollars) is given as a function of the amount of money a (in thousands of dollars) they spend on advertising by $R = f(a)$. Suppose that f is invertible.

a. [2 points] Which of the following is a valid interpretation of the equation $(f^{-1})'(75) = 0.5$? Circle one option.

- If the company spends \$75,000 more on advertising, their revenue will increase by about \$500.
- If the company increases their advertising expenditure from \$75,000 to \$76,000, their revenue will increase by about \$500.
- If the company wants a revenue of \$75,000, they should spend about \$500 on advertising.
- If the company wants to increase their revenue from \$75,000 to \$76,000, they should spend about \$500 more on advertising.

Solution: The last option.

b. [2 points] The company plans to spend about \$100,000 on advertising. If $f'(100) = 0.5$, should the company spend more or less than \$100,000 on advertising? Justify your answer.

Solution:
They should spend less on advertising, because if they increase their advertising expenditure by \$1000, they will only gain about \$500 in revenue.

c. [5 points] The company's financial advisor claims that he has a formula for the dependence of revenue on advertising expenditure, and it is

$$f(a) = a \ln(a + 1).$$

Using this formula, write the *limit definition* of $f'(100)$. You do not need to simplify or evaluate.

Solution:

$$f'(100) = \lim_{h \to 0} \frac{(100 + h) \ln(100 + h + 1) - 100 \ln(101)}{h}$$

9. [10 points] On the axes below, sketch a well-labeled graph of a function $f(x)$, defined for all x, satisfying the following properties:

- $f(0) = 0$.
- $\lim_{x\to 1} f(x)$ exists but $f(x)$ is not continuous at $x = 1$.
- $f'(x)$ is increasing on the interval $(3, 5)$.
- $f''(x)$ changes sign at $x = -3$.
- $f'(4) > 0$.
- $f(x) = f(x + 5)$ for all x.

You need only show the graph on the domain $[-5, 5]$.