Math 115 — First Midterm

February 12, 2013

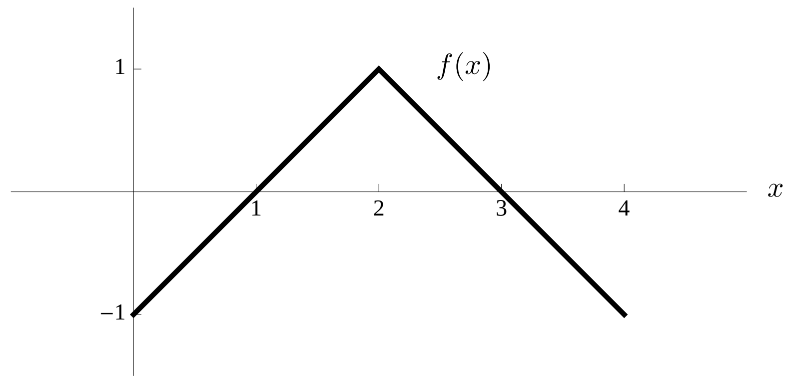
Name: EXAM SOLUTIONS

Instructor: ____________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	6	
3	14	
4	10	
5	10	
6	13	
7	15	
8	10	
9	7	
10	5	
Total	100	

1. [10 points] Suppose $g(x) = x^2$. The graph of a function $f(x)$ is given below. For parts (a)-(c) below, write *all* real numbers z that make the statement true. If no values of z make the statement true, write "NONE". You do not need to show your work.



a. [2 points] $f(g(z)) = 1$.

$z =$ $\mathbf{-\sqrt{2}, \sqrt{2}}$

Solution: We need $f(z^2)$ to be 1, so we need $z^2 = 2$. The two possibilities are $z = \pm\sqrt{2}$.

b. [2 points] $g(f(z)) = 0$.

$z =$ **1,3**

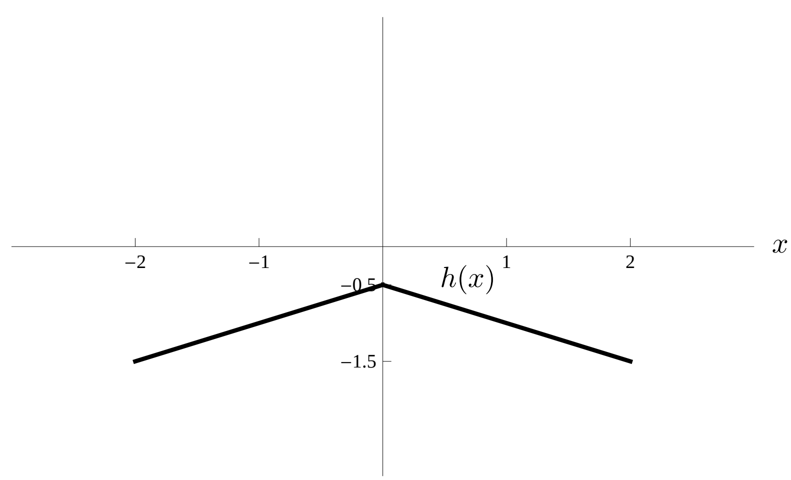
Solution: We need $f(z)^2$ to be 0, so we need $f(z)$ to be 0. The two possibilities are $z = 1$ or $z = 3$.

c. [2 points] $f(f(z)) = 0$.

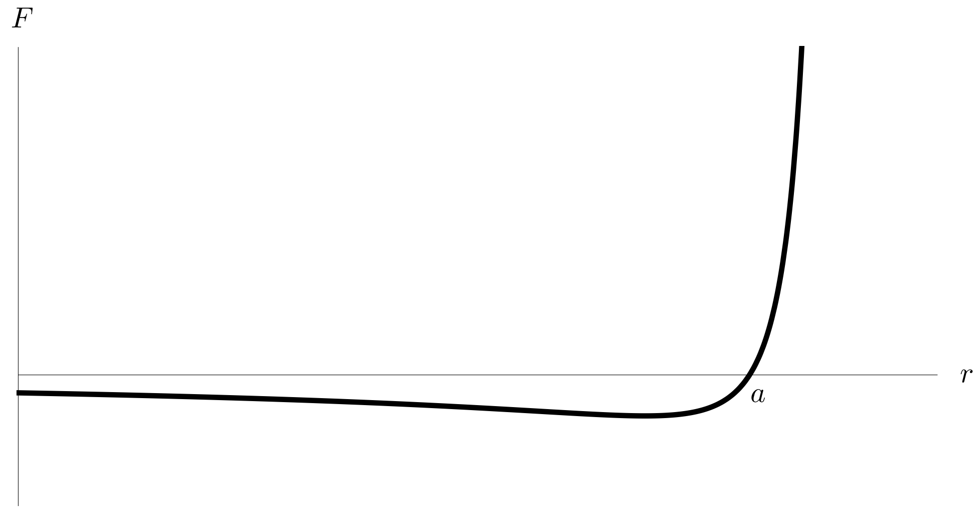
$z =$ **2**

Solution: We need $f(z)$ to be 1 or 3. The only possibility is $z = 2$.

d. [4 points] The function $h(x)$ is given by the formula $h(x) = \frac{1}{2}f(x+2) - 1$. On the axes provided below, draw a well-labeled graph of $h(x)$.



2. [6 points] The force, F, between two magnets arranged in an array depends on the distance r separating them. Looking at the graph below, a positive F represents a repulsive force; a negative F represents an attractive force. The horizontal intercept of the graph is $r = a$.



a. [1 point] What happens to the force if the magnets start with $r = a$ and are pulled slightly farther apart?

Solution: The force increases (or: the magnets repel each other, the force becomes more repulsive, the force becomes more positive...).

b. [1 point] What happens to the force if the magnets start with $r = a$ and are pushed slightly closer together?

Solution: The force decreases (or: the magnets attract each other, the force becomes more attractive, the force becomes more negative...).

c. [4 points] The magnets are said to be in *stable equilibrium* if the force between them is zero and the magnets tend to return to the equilibrium after a minor disturbance. Does $r = a$ represent a stable equilibrium? Give a brief explanation.

Solution: They are not in stable equilibrium. It's true that the force between them is 0 when $r = a$, but if they are pulled slightly apart, they will tend to move still farther apart, and if they are pushed closer together, they will tend to move still closer together.

3. [14 points] Laura and Eddie are co-owners of a caffeinated soap factory. Let $M(x)$ denote the mass, in grams, of caffeine in a bar of soap that causes a typical customer's bloodstream caffeine content to be x mg.

a. [4 points] Assuming that $M(x)$ is an invertible function, give a practical interpretation of the statement $M^{-1}(2) = 12$.

Solution: If a bar of soap contains 2 grams of caffeine, it will cause a typical caffeine customer's bloodstream content to be 12 mg.

b. [4 points] Under the same assumption, give a practical interpretation of the statement $M'(13) = 0.7$.

Solution: If we compare a bar of soap that causes a typical customer's bloodstream caffeine content to be 13 mg with one that causes a typical customer's bloodstream caffeine content to be 14 mg, we expect the second bar to contain approximately .7 more grams of caffeine.

c. [6 points]

Laura and Eddie know that $M(x)$ is either a linear or an exponential function, but they aren't sure which. From experimenting, they know that $M(12) = 2$ and $M(14) = 4$. They need more data to determine which is correct. For each of the following hypothetical experimental results, circle EXPONENTIAL if the result shows that $M(x)$ could be exponential, circle LINEAR if the result shows that $M(x)$ could be linear, or circle EITHER if the result does not rule out either possibility. Assume Laura and Eddie's equipment gives experimental evidence which is accurate to within .1 mg.

i. Laura and Eddie discover that $M(x)$ is an invertible function.

EXPONENTIAL LINEAR $\boxed{\text{EITHER}}$

ii. Laura and Eddie discover that $M'(17.2) = M'(18.3)$.

EXPONENTIAL $\boxed{\text{LINEAR}}$ EITHER

iii. Laura and Eddie discover that when there are 7 grams of caffeine in the soap, the caffeine level in a typical customer's bloodstream is roughly 15.6 mg.

$\boxed{\text{EXPONENTIAL}}$ LINEAR EITHER

4. [10 points] A motorcyclist heads north from an intersection after a stoplight turns green. The table below records the data on the motorcyclist's speedometer, measuring her velocity, $v(t)$, in feet per second, t seconds after the stoplight turns green. Assume that the motorcyclist does not slow down at any point during the interval of time we are measuring.

t	0	2	4	6
$v(t)$	0	5	15	30

a. [3 points] Recall that the acceleration function, $a(t)$, is the derivative of the velocity function. Use the table to estimate $a(2)$. Include units.

Solution: We have $a(2) \approx \frac{v(4)-v(2)}{2} = 5$ feet per second per second. Alternatively, $a(2) \approx \frac{v(2)-v(0)}{2-0} = 2.5$ feet per second per second, or we could take the average of these two estimates, 3.75 feet per second per second (or ft/s^2).

b. [3 points] The "jerk" $j(t)$ of the motorcycle is the derivative of the acceleration function. Use the table to estimate $j(2)$. Include units.

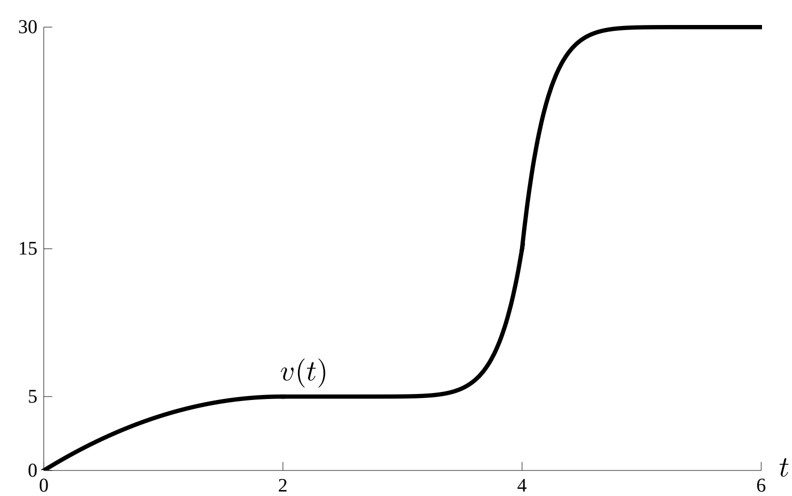
Solution: We can fill in the table with new approximations for $a(t)$ using the same logic as in the previous problem. (Your approximations may be different.)

t	0	2	4
$v(t)$	0	5	15
$a(t) \approx$	2.5	5	7.5

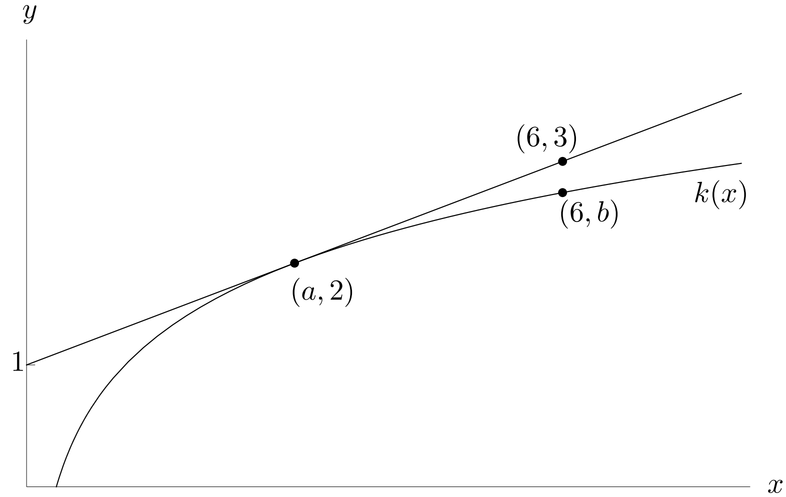
Now $j(2) \approx \frac{a(4)-a(2)}{4-2} \approx 1.25$ feet per second per second per second (or ft/s^3). If you used different estimates for $a(t)$, your answer could be different.

c. [4 points] Given everything we know about the motorcyclist, can we definitely conclude that $a(4) \leq 8$? If you answer YES, then explain your reasoning. If you answer NO, then sketch a graph of a velocity function $v(t)$ which is consistent with all the information in this problem, but which has $a(4) > 8$.

Solution: No, we cannot conclude that. The graph of $v(t)$ might look like this:



5. [10 points] The figure below shows the graph a function $k(x)$ and its tangent line at a point $(a, 2)$. The average rate of change of $k(x)$ between $x = a$ and $x = 6$ is $1/6$.



Find **exact** numerical values for the following. If it is not possible to find a value, write "NP". You do not need to show your work.

a. [2 points] $a =$ **3**

Solution: The slope of the tangent line is $\frac{1}{3}$ and its y-intercept is 1, so its equation is $y = \frac{1}{3}x + 1$. Since $(a, 2)$ lies on the tangent line, $a = 3$.

b. [2 points] $b =$ **5/2**

Solution: We know that the average rate of change between $x = a$ and $x = 6$ is $1/6$. This is the slope of the secant line connecting $(3, 2)$ and $(6, b)$. After some algebra we obtain $b = \frac{5}{2}$.

c. [2 points] $k'(2) =$ **NP**

Solution: We cannot find $k'(2)$ (the y-coordinate of the given point is 2, not the x-coordinate).

d. [2 points] $k'(a) =$ **1/3**

Solution: This is the slope of the tangent line, which is $\frac{1}{3}$.

e. [2 points] $k'(6) =$ **NP**

Solution: We cannot find $k'(6)$ since the given line is not tangent to the graph when $x = 6$ (and the statement about average change refers to a secant line, not a tangent line).

6. [13 points] Suppose $n(x) = (x + \frac{1}{2})e^x$.

a. [4 points] Using the limit definition of the derivative, write an explicit expression for $n'(2)$. Your expression should not contain the letter "n". Do not try to evaluate your expression.

Solution:

$$n'(2) = \lim_{h \to 0} \frac{(2 + h + \frac{1}{2})e^{2+h} - (2 + \frac{1}{2})e^2}{h}$$

The derivative of $n(x)$ is $n'(x) = (x + \frac{3}{2})e^x$.

b. [3 points] Using the given formula for $n'(x)$, write an equation for the tangent line to the graph of $n(x)$ at $x = 2$.

Solution: The slope is $(2 + \frac{3}{2})e^2$, and the y-coordinate when $x = 2$ is $(2 + \frac{1}{2})e^2$, so using the point-slope formula for a line, we get the equation

$$y - (2 + \frac{1}{2})e^2 = (2 + \frac{3}{2})e^2(x - 2)$$

c. [3 points] Write an equation for the tangent line to the graph of $n(x)$ at $x = a$ where a is an unknown constant.

Solution: Using the same logic, we get

$$y - (a + \frac{1}{2})e^a = (a + \frac{3}{2})e^a(x - a)$$

d. [3 points] Using your answer from (c), find a value of a so that the tangent line to the graph of $n(x)$ at $x = a$ passes through the origin.

Solution: We want the line from the previous part of the problem to pass through $(0, 0)$, so we have the equation

$$-(a + \frac{1}{2})e^a = (a + \frac{3}{2})e^a(-a).$$

After dividing out the term e^a, this becomes a quadratic equation in a:

$$a^2 + \frac{1}{2}a - \frac{1}{2} = 0$$

Factoring or using the quadratic formula, we conclude that $a = -1$ or $a = \frac{1}{2}$.

7. [15 points] In each of the following problems, give a *formula* for a function whose domain is all real numbers, with *all* of the indicated properties. If there is no such function, then write "NO SUCH FUNCTION EXISTS". You do not need to show your work.

a. [6 points] A sinusoidal function $P(t)$ with the following three properties:

(i.) The period of the graph of $P(t)$ is 7.

(ii.) The graph of $P(t)$ attains a maximum value at the point $(1, 20)$.

(iii.) The graph of $P(t)$ attains a minimum value at the point $(-2.5, -6)$.

Solution: If we move the function to the left by 1, we get a cosine function with period 7, amplitude 13, and vertical shift 7. Call this shifted function $\widetilde{P}(t)$. Since

$$\widetilde{P}(t) = 13\cos\left(\frac{2\pi}{7}t\right) + 7$$

and $P(t)$ is $\widetilde{P}(t)$ shifted right by one, we get

$$P(t) = 13\cos\left(\frac{2\pi}{7}(t-1)\right) + 7$$

b. [3 points] A function $h(x)$ with the following two properties:

(i.) $h(x)$ is concave down for all x

(ii.) $h(x) > 0$ for all x.

Solution: No such function exists. If the function is decreasing at some point, it will decrease faster and faster until it touches the x-axis. If the function is increasing at some point, similar logic applies (reading right to left, rather than left to right). The only other possibility is that the function is not increasing or decreasing anywhere, but then it would just be a horizontal line with no concavity.

c. [3 points] A function $j(x)$ with the following two properties:

(i.) $j(x)$ is decreasing for all x.

(ii.) $j(x)$ is concave up for all x.

Solution: One example of such a function that we have encountered is an exponential decay function, for instance $j(x) = e^{-x}$.

d. [3 points] A *rational* function $\ell(x)$ with the following two properties:

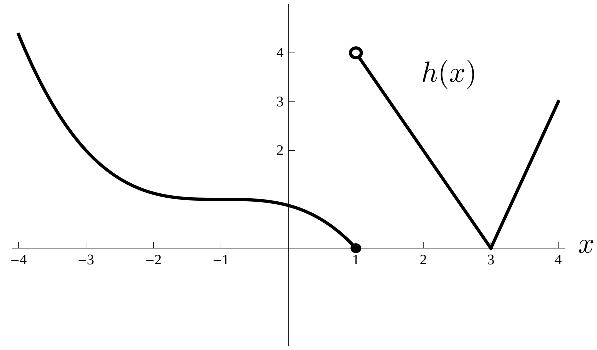
(i.) $\ell(0) = 2$.

(ii.) The line $y = 2$ is a horizontal asymptote to the graph of $\ell(x)$.

Solution: We want the function to be defined everywhere, so we need the denominator not to have any roots. For instance, we can take the polynomial $x^2 + 1$ for the denominator. If we make this choice, the numerator will need to have degree 2 (so that the horizontal asymptote exists and is not the line $y = 0$) and first coefficient $2x^2$. Now we rig the numerator in whatever way we like to get $\ell(0) = 2$. One answer is

$$\ell(x) = \frac{2x^2 + 2x + 2}{x^2 + 1}$$

8. [10 points] The graph of a function $h(x)$ is given below.



a. [1 point] List all x-values with $-4 < x < 4$ where $h(x)$ is not continuous. If there are none, write NONE.

Solution:

$$x = 1$$

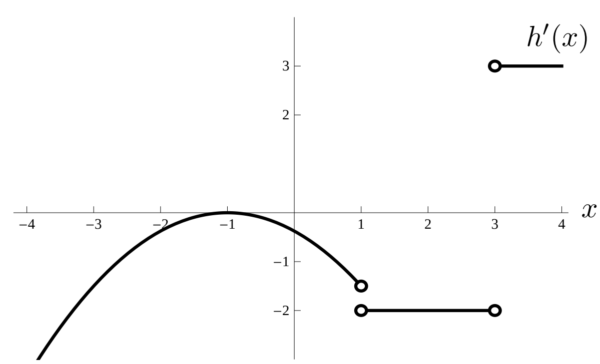
b. [1 point] List all x-values with $-4 < x < 4$ where $h(x)$ is not differentiable. If there are none, write NONE.

Solution:

$$x = 1, 3$$

c. [8 points] On the axes provided, carefully draw a graph of $h'(x)$. Be sure to label important points or features on your graph.

Solution:



9. [7 points] The air in a factory is being filtered so that the quantity of a pollutant, P (in mg/liter), is decreasing exponentially. Suppose t is the time in hours since the factory began filtering the air. Also assume 20% of the pollutant is removed in the first five hours.

a. [2 points] What percentage of the pollutant is left after 10 hours?

Solution: Since 80% of the pollution is left after 5 hours, we see that 80% of 80% of the pollution will be left after 10 hours. This is 64%.

b. [5 points] How long is it before the pollution is reduced by 50%?

Solution: Using the equation

$$P(t) = P_0 a^t,$$

and plugging in the known point $(5, .8P_0)$, we see that

$$.8P_0 = P_0 a^5.$$

So $a = (.8)^{\frac{1}{5}}$ and the formula is

$$P(t) = P_0(.8)^{\frac{t}{5}}.$$

We could also have gotten this formula without algebra, using the definition of decay rate and converting from intervals of 5 hours to intervals of hours.
We want to find a value of t such that $P(t) = .5P_0$. Algebraically, this says that

$$.5P_0 = P_0(.8)^{\frac{t}{5}}.$$

Dividing out P_0 and taking logarithms, we get

$$\ln(.5) = \frac{t}{5}\ln(.8),$$

so $t = 5\frac{\ln(.5)}{\ln(.8)} \approx 15.53$ hours. We could also solve this problem with the initial equation

$$P(t) = P_0 e^{kt}.$$

If you did it that way, you should have gotten $k \approx -.0446$.

10. [5 points] Define a function

$$f(x) = \begin{cases} \dfrac{-x^3 + 5x^2}{x - 5} & x \neq 5, \\ k & x = 5. \end{cases}$$

a. [3 points] Find a value of k so that $f(x)$ is a continuous function for all real numbers x.

Solution: The formula for $f(x)$ when $x \neq 5$ simplifies algebraically to $-x^2$. Since we want $f(x)$ to be continuous when $x = 5$, we need

$$k = \lim_{x \to 5} f(x) = -(5^2) = -25.$$

b. [2 points] For the value of k you found, is $f(x)$ differentiable at $x = 5$? Briefly explain.

Solution: Yes. If $k = -25$, there are no sharp corners or vertical tangents – in fact, the function $f(x)$ is identical to the function $-x^2$, which we know to be differentiable.