Math 115 — Final Exam

April 26, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	5	
3	16	
4	6	
5	15	
6	11	
7	11	
8	12	
9	15	
Total	100	

1. [9 points] Let $g(x) = x + ke^x$, where k is any constant.

a. [4 points] Write an explicit expression for the limit definition for the derivative of $g(x)$ at $x = 2$. Your expression should not include the letter 'g'. Do not evaluate your expression.

Solution:

$$g'(2) = \lim_{h\to 0} \frac{(2 + h + ke^{2+h}) - (2 + ke^2)}{h}$$

b. [5 points] Find all values of k for which the function $g(x)$ has a critical point. Do not try to use your answer from (a).

Solution: The function $g(x)$ has a critical point $\boxed{\text{whenever } k < 0.}$ Since $g'(x) = 1 + ke^x$, the only possible critical point is when $1 + ke^x = 0$, or

$$e^x = \frac{-1}{k}.$$

This equation will have no solutions if $k \geq 0$, but will have a solution for any value of k which is negative.

2. [5 points] A piece of wire of length L is cut into two pieces. One piece of length x cm is made into a circle and the rest is made into a square. Write an expression for the sum of the areas, A, of the circle and square in terms of the length L and the variable x. Do not optimize A.

Solution: The part of the wire that will be made into a circle has length x, so the circumference of that circle will be x. This means that $x = 2\pi r$, where r denotes the radius of the circle, so $r = \frac{x}{2\pi}$. The area of the circle will be $\pi r^2 = \pi \left(\frac{x}{2\pi}\right)^2$.
The part of the wire that will be made into a square has length $L - x$, so the perimeter of that square will be $L - x$. This means that $L - x = 4s$, where s denotes the length of a side of the square, so $s = \frac{L-x}{4}$. The area of the square will be $s^2 = \left(\frac{L-x}{4}\right)^2$. Thus the final answer is

$$A = \pi \left(\frac{x}{2\pi}\right)^2 + \left(\frac{L - x}{4}\right)^2$$

3. [16 points] Eddie and Laura have signed an exclusive contract to begin producing the world's first caffeinated soup, called Minestromnia. If they charge \$4.00 per liter or more for the soup, then nobody will buy it. Otherwise, if they charge p dollars per liter for the soup, they will sell $g(p)$ liters, where

$$g(p) = 500(16 - p^2).$$

a. [3 points] Write an expression for the revenue $R(p)$ that Eddie and Laura will generate if they charge p dollars per liter of soup.

Solution: The revenue is the price times the number of liters sold, so

$$R(p) = pg(p) = 500p(16 - p^2)$$

(To be really precise, we should say

$$R(p) = \begin{cases} 500p(16 - p^2), & 0 \leq p < 4 \\ 0, & p \geq 4 \end{cases}$$

or something like this.)

b. [3 points] The ingredients in a liter of Minestromnia cost \$1.00. To start their business, Eddie and Laura need to purchase a very large soup kettle and other equipment at a total cost of \$700.00. Write an expression for the total cost $C(p)$, including fixed costs, of producing $g(p)$ liters of soup.

Solution: The fixed costs are 700 and each liter costs a dollar, for a total cost of

$$C(p) = 700 + g(p) = 700 + 500(16 - p^2).$$

c. [6 points] What price should Eddie and Laura charge per liter of Minestromnia in order to maximize their profits? Be sure to explain how you know that this price produces the maximum possible profit.

Solution: Write $\pi(p) = R(p) - C(p)$. Then after some algebra, we get $\pi(p) = -500p^3 + 500p^2 + 8000p - 8700$, so

$$\pi'(p) = -1500p^2 + 1000p + 8000$$

Setting $\pi'(p)$ equal to zero gives a quadratic equation, whose solutions are $2\frac{2}{3} \approx 2.67$ and the illogical -2. So the critical point is $p = 2.67$. Plugging in the critical point gives

$$\pi(2.67) \approx 6707.$$

The reasonable prices that Eddie and Laura can set lie in a closed interval: $0 \leq p \leq 5$. The profit is clearly negative at both endpoints (they aren't getting any revenue at either endpoint) so the maximum profit occurs when the price is approximately \$2.67.

3. (continued)

d. [4 points] Give a practical interpretation of the formula

$$g'(3.5) = -3500$$

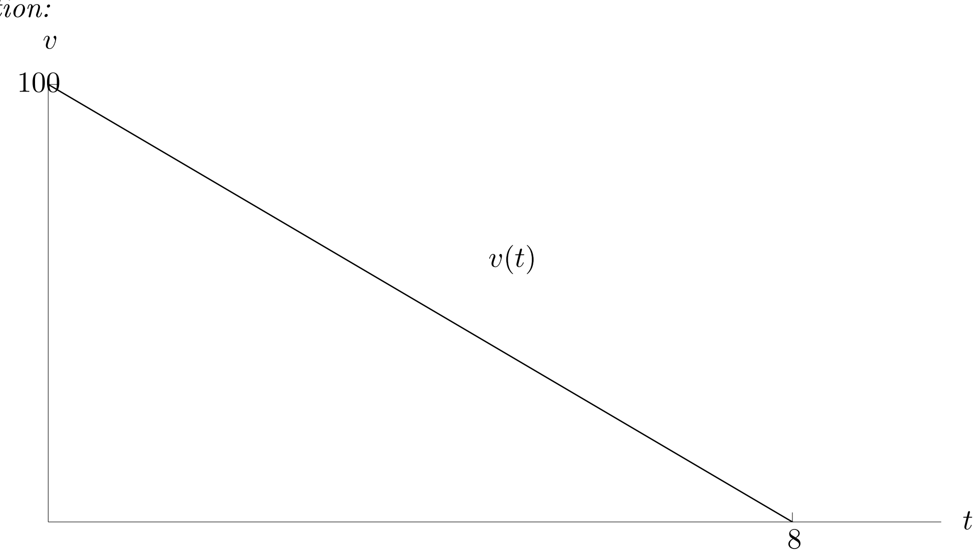
that begins with
"If Eddie and Laura decrease the price of the soup from \$3.50 per liter to \$3.40 per liter ..."

Solution: ... they expect to sell about 350 more liters of Minestromnia.

4. [6 points] A car, initially going 100 feet per second, brakes at a constant rate (constant negative acceleration), coming to a stop in 8 seconds. Let t be the time in seconds after the car started to brake.

a. [3 points] Sketch a graph of the velocity of the car from $t = 0$ to $t = 8$, being sure to include labels.

Solution:



b. [3 points] Exactly how far does the car travel? Make it clear how you obtained your answer.

Solution: It's $\int_0^8 v(t)dt$, which is the area of the triangle above: $\frac{1}{2} * 100 * 8 = 400$ ft.

5. [15 points] The function $h(x)$ is not known, but the *derivative* of $h(x)$ is given by the formula

$$h'(x) = \sin(x)e^{x^2+1}.$$

a. [2 points] Find a formula for $h''(x)$.

Solution:

$$h'(x) = \cos(x)e^{x^2+1} + 2x\sin(x)e^{x^2+1}$$

b. [6 points] List all critical points for $h(x)$ in the open interval $-2\pi < x < 2\pi$. For each point, use an appropriate test to determine whether it is a local maximum, local minimum, or neither.

Solution: A critical point of $h(x)$ occurs when $h'(x) = 0$. Since $e^{x^2+1} \neq 0$, this is when $\sin(x) = 0$, so $x = -\pi, 0, \pi$. Plugging these values into the second derivative gives

x	$g''(x)$
$-\pi$	$-e^{-\pi^2+1} < 0$
0	$e > 0$
π	$-e^{-\pi^2+1} < 0$

So $x = -\pi$ is a local maximum, $x = 0$ is a local minimum, and $x = \pi$ is a local maximum.

c. [2 points] For which x-value in the closed interval $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$ does $h(x)$ attain its maximum value? (Do not try to find the y-coordinate.)

Solution: The function $h'(x)$ is non-negative in this interval. Therefore $h(x)$ is increasing and attains its maximum value at the right endpoint $x = \frac{\pi}{2}$.

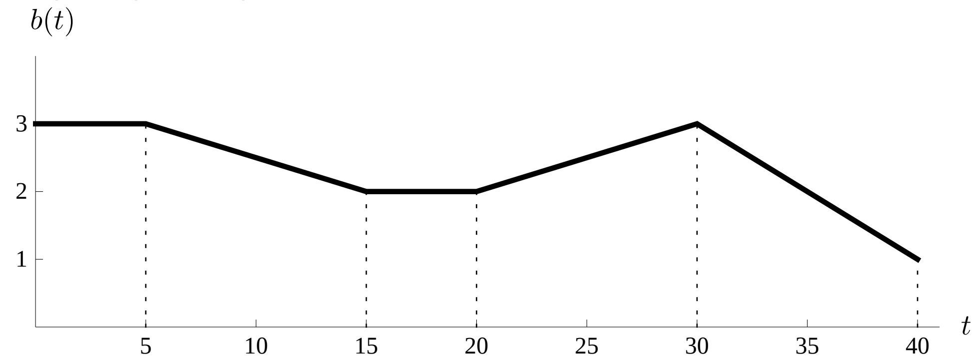
d. [5 points] Write out all terms for a right-hand Riemann sum with three subintervals which approximates

$$\int_0^1 \sin(x)e^{x^2+1}dx.$$

Solution:

$$\frac{1}{3}\left(\sin\left(\frac{1}{3}\right)e^{\left(\frac{1}{3}\right)^2+1} + \sin\left(\frac{2}{3}\right)e^{\left(\frac{2}{3}\right)^2+1} + \sin(1)e^2\right)$$

6. [11 points] During a recent practice, the UM basketball team was split up into a Maize team and Blue team. The teams played against each other for a full 40 minute game. The graph of the function $b(t)$ below shows the Blue team's scoring rate in points per minute t minutes after the beginning of the game.



The Maize team scored 2 points per minute for the first 30 minutes of the game and then they scored 3 points per minute for the last 10 minutes.

a. [3 points] Calculate the exact value of the integral

$$\int_5^{15} (b(t) - 2)dt.$$

Solution: From the picture we see that the area of the triangle between the graph and the line $y = 2$ is 5.

b. [3 points] Give a practical interpretation of the integral from (a) in the context of the problem.

Solution: Between the fifth and fifteenth minute of the game, the Blue Team's scored 5 points more than the Maize team.

c. [3 points] Calculate the average scoring rate of the Blue team during the first half of the game. Include units in your answer.

Solution:

$$\frac{1}{20}\int_0^{20} b(t)dt = \frac{5}{2} \text{ points/minute}$$

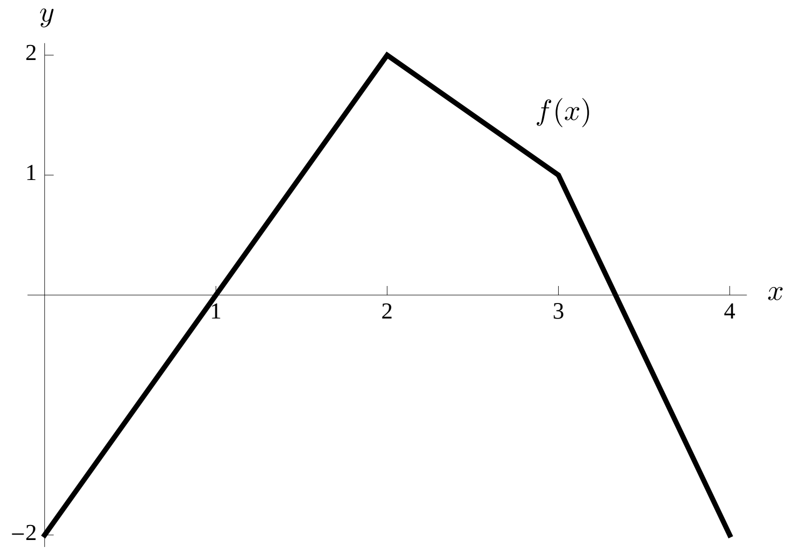
d. [2 points] What was the final score of the game? You do not need to show your work.

Blue Team's Score= **95**

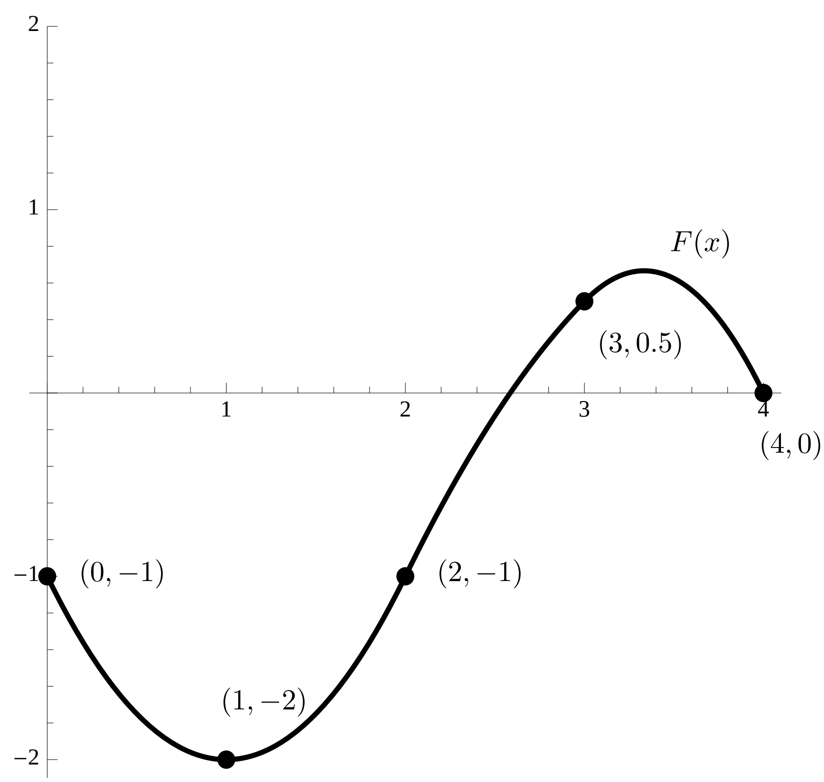
Maize Team's Score= **90**

7. [11 points] Below is a picture of the function $f(x)$, which is piecewise linear. On the axes provided, sketch an *antiderivative*, $F(x)$, of the function $f(x)$, with $F(2) = -1$. To receive full credit:

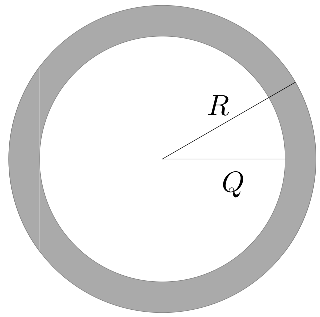
- Label the points on the graph of $F(x)$ where $x = 0, 1, 2, 3, 4$, including y-coordinates.
- Be sure the concavity and local extrema on the graph of $F(x)$ are clear.



Solution:



8. [12 points] Mitch puts a thin metal ring in an oven. A picture of the ring, which is made by removing a solid metal circular region of radius Q cm from a solid metal circular region of radius R cm, is below. The circles have the same center.



The ring expands as the temperature gets hotter, and so R and Q are each functions of the time, t, measured in minutes since Mitch put the ring into the oven. The following table gives some values for the functions R and Q, as well as their derivatives.

t	19	20	21
$R(t)$	1.95	2	2.06
$Q(t)$	1.8	1.75	1.68
$R'(t)$	.04	.05	.05
$Q'(t)$	−.06	−.06	−.04

a. [2 points] Assuming that $R(t)$ is an invertible function, compute

$$(R^{-1})'(2.06).$$

Do not give an approximation.

Solution: We have

$$(R^{-1})'(2.06) = \frac{1}{R'(R^{-1}(2.06))} = \frac{1}{.05} = 20 \text{ minutes/cm}$$

b. [2 points] Compute the exact value of

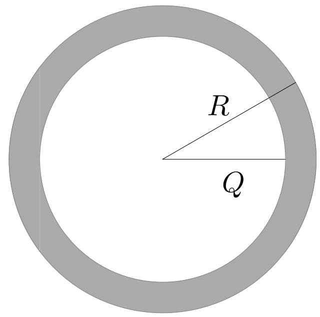
$$\int_{19}^{21} Q'(t)dt.$$

Do not give an approximation.

Solution: By the fundamental theorem of calculus $\int_{19}^{21} Q'(t)dt = Q(21) - Q(19) = -.12$

8. (continued)

The figure and table below are reproduced from the previous page, in case you need them on this page.



t	19	20	21
$R(t)$	1.95	2	2.06
$Q(t)$	1.8	1.75	1.68
$R'(t)$	.04	.05	.05
$Q'(t)$	$-.06$	$-.06$	$-.04$

c. [2 points] Write an expression for $A(t)$, the area of the ring t minutes after Mitch put it in the oven, in terms of $R(t)$ and $Q(t)$.

Solution:

$$A(t) = \pi(R(t))^2 - \pi(Q(t))^2$$

d. [6 points] How fast is the area of the ring growing 20 minutes after Mitch puts the ring in the oven? Include units in your answer.

Solution: By the chain rule,

$$A'(t) = 2\pi R(t)R'(t) - 2\pi Q(t)Q'(t),$$

so

$$A'(20) = 2\pi * 2 * .05 - 2\pi * 1.75 * (-.06) = .41\pi \frac{\text{cm}^2}{\text{min}}$$

9. [15 points] For each question below, there is only one correct answer. Circle exactly one answer. Unclear answers will receive no credit. There is no penalty for guessing.

a. [3 points] A company's maximum profit is earned when it produces $q = 20$ goods. If its *marginal cost* function is given by

$$MC(q) = 7q,$$

which of the following could be the company's *revenue* function?

A. $R(q) = q + 120$

B. $R(q) = 7$

C. $R(q) = q^7 + 2$

D. $\boxed{R(q) = 2q^2 + 60q}$

E. $R(q) = \frac{1}{20}q^7 + 2$

b. [3 points] The number ℓ is a positive constant. Which of the following numbers is the maximum value of the function $f(x) = (x - \ell)^3 + 12\ell^3$ on the closed interval $[-\ell, 2\ell]$? (These numbers are y-values, not x-values).

A. $11\ell^3$

B. $20\ell^3$

C. $\boxed{13\ell^3}$

D. $4\ell^3$

E. $12\ell^3$

c. [3 points] The number p is a constant. Which of the following functions is an antiderivative of $g(x) = \ln(x + p)$?

A. $G(x) = \frac{p}{x+p}$

B. $G(x) = \frac{1}{x+p}$

C. $\boxed{G(x) = (x + p)(\ln(x + p)) - x}$

D. $G(x) = \frac{\ln(x+p)}{p} - x$

E. $G(x) = x^2 \ln(x + p) - x$

d. [3 points] Suppose $g'(x) > 0$ on the interval $[3, 5]$, $g(3) = 12$, and $g(5) = 20$. We want to use a Riemann sum with equal-size subdivisions to approximate

$$\int_3^5 g(x)dx,$$

If we want to guarantee that the error in our estimate is no larger than 1/4, then what is the minimum number of subdivisions that we must use?

A. 8

B. 16

C. 32

D. $\boxed{64}$

E. We cannot guarantee this much accuracy, no matter how many subdivisions we use.

e. [3 points] If

$$\int_{-1}^4 (2f(x) - 7)dx = -31,$$

then which of the following values is equal to

$$\int_{-1}^4 f(x)dx?$$

A. -24

B. -12

C. $\boxed{2}$

D. 4

E. 31