On my honor, as a student, I have neither given nor received unauthorized aid on this academic work. **Signed:** ______________________

Math 115 — Final Exam

April 28, 2014

Name: ______________________

Instructor: ______________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	10	
3	11	
4	14	
5	9	
6	8	
7	7	
8	11	
9	9	
10	11	
Total	100	

1. [10 points] The table below gives several values of a function $f(x)$ and its derivative. Assume that both $f(x)$ and $f'(x)$ are defined and differentiable for all x.

x	0	1	2	3	4	5	6
$f(x)$	0	3	4	2	-1	-3	5
$f'(x)$	4	2	-1	-5	-2	7	9
$f''(x)$	-1	-3	-5	0	4	3	1

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE

a. [2 points] Find $\displaystyle\int_0^4 f''(x)\,dx$.

Answer: $\displaystyle\int_0^4 f''(x)\,dx =$ ________________

b. [2 points] Find $\displaystyle\int_2^5 (3f(x)+1)\,dx$.

Answer: $\displaystyle\int_2^5 (3f(x)+1)\,dx =$ ________________

c. [3 points] Find the average value of $4f'(x)+x$ on the interval $[1,6]$.

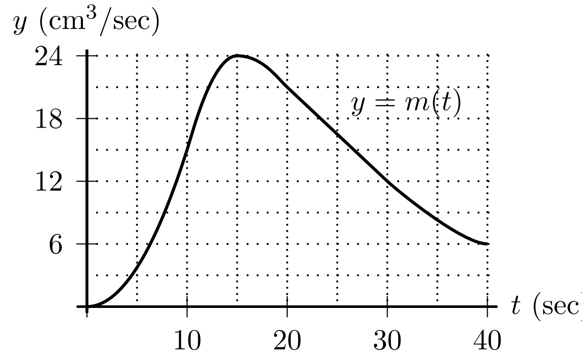
Answer: ________________

d. [3 points] Assuming that $f(x)$ is an odd function, find $\displaystyle\int_{-3}^3 f(x)\,dx$ and $\displaystyle\int_{-3}^3 f'(x)\,dx$.

Answer: $\displaystyle\int_{-3}^3 f(x)\,dx =$ ________________ and $\displaystyle\int_{-3}^3 f'(x)\,dx =$ ________________

2. [10 points]

Kathy puts a very large marshmallow in the microwave for forty seconds and watches as it inflates. Let $m(t)$ be the rate of change of the volume of the marshmallow, in cm^3/sec, t seconds after Kathy puts it in the microwave. The graph of $y = m(t)$ is shown to the right.



a. [2 points] Write a definite integral equal to the total change in volume, in cm^3, of the marshmallow while in the microwave. (You do not need to evaluate the integral.)

Answer: ________________________

b. [3 points] Estimate your integral from part (a) using a right-hand sum with $\Delta t = 10$. Be sure to write out all of the terms in the sum.

Answer: ________________________

c. [5 points] Assume that throughout its time in the microwave, the marshmallow is a cylinder. After 30 seconds in the microwave, the marshmallow is a cylinder with radius 4.5 cm and height 11 cm. At that moment, the height is increasing at 0.08 cm/sec. How fast is the radius of the marshmallow increasing at that moment?
Recall that the volume V of a cylinder of radius r and height h is $V = \pi r^2 h$, and remember to include units.

Answer: ________________________

3. [11 points] For positive constants a and b, the potential energy of a particle is given by

$$U(x) = a\left(\frac{5b^2}{x^2} - \frac{3b}{x}\right).$$

Assume that the domain of $U(x)$ is the interval $(0, \infty)$.

a. [2 points] Find the asymptotes of $U(x)$. If there are none of a particular type, write NONE.

Answer: Vertical asymptote(s): __________ Horizontal asymptote(s): __________

b. [6 points] Find the x-coordinates of all local maxima and minima of $U(x)$ in the domain $(0, \infty)$. If there are none of a particular type, write NONE. You must use calculus to find and justify your answers. Be sure to provide enough evidence to justify your answers fully.

Answer: Local max(es) at $x =$ __________ Local min(s) at $x =$ __________

c. [3 points] Suppose $U(x)$ has an inflection point at $(6, -14)$. Find the values of a and b. *Show your work, but you do not need to verify that this point is an inflection point.*

Answer: $a =$ __________ and $b =$ __________

4. [14 points] One of the ways Captain Christina likes to relax in her retirement is to go for long walks around her neighborhood. She has noticed that early every Tuesday morning, a truck delivers butter to a local bakery famous for its cookie dough. Consider the following functions:

- Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of butter.
- Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter.
- Let $u(t)$ be the instantaneous rate, in pounds per hour, at which butter is being unloaded t hours after 4 am.

Assume that C, K, and u are invertible and differentiable.

a. [2 points] Interpret $K(C^{-1}(10)) = 20$ in the context of this problem.
Use a complete sentence and include units.

b. [3 points] Interpret $\int_5^{12} K'(b)\, db = 40$ in the context of this problem.
Use a complete sentence and include units.

c. [3 points] Give a single mathematical equality involving the <u>derivative</u> of C which supports the following claim:
It costs the bakery approximately \$0.70 less to buy 14.8 pounds of butter than to buy 15 pounds of butter.

Answer: ____________________

d. [3 points] Give a single mathematical equality which expresses the following claim:
The number of pounds of butter unloaded between 5 and 8 am is twice as many as the bakery needs to make 5000 cups of cookie dough.

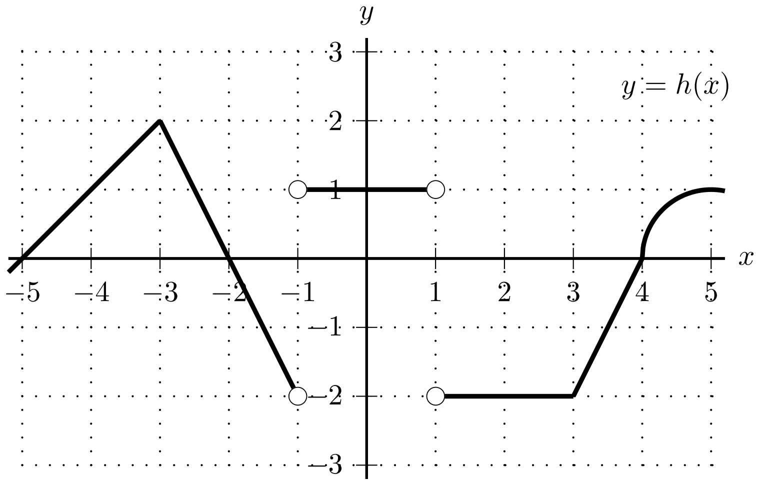
Answer: ____________________

e. [3 points] Assume that $u(t) > 0$ and $u'(t) < 0$ for $0 \le t \le 4$ and that $u(2) = 800$.
Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-IV.

I. 0 II. 800 III. $\int_1^2 u(t)\, dt$ IV. $\int_2^3 u(t)\, dt$

________ $<$ ________ $<$ ________ $<$ ________

5. [9 points] The graph of a portion of $y = h(x)$ is shown below.



Note: The portion of the graph of $h(x)$ between $x = 4$ and $x = 5$ is part of a circle of radius 1 centered at the point $(5, 0)$.

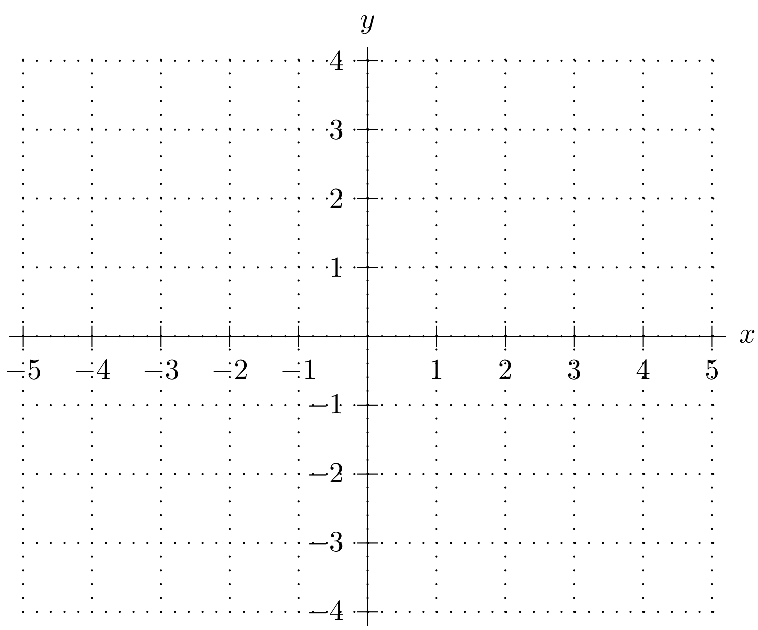
Let $H(x)$ be the continuous antiderivative of $h(x)$ with $H(0) = 2$.

a. Complete the following table with the exact values of $H(x)$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$H(x)$						2					

b. On the axes below, sketch the graph of $y = H(x)$. Be sure that you pay close attention to each of the following:

- where $H(x)$ is and is not differentiable
- the values of $H(x)$ from the table above
- the sign of $H(x)$, where $H(x)$ is increasing/decreasing/constant, and the concavity of $H(x)$



6. [8 points] Suppose that a ring made entirely of gold and platinum is made from g ounces of gold and p ounces of platinum and that gold costs h dollars per ounce and platinum costs k dollars per ounce. Then the value, in dollars, of the ring is given by

$$v = gh + pk.$$

a. [3 points] Pat has a ring made entirely of gold and platinum. Pat's ring is made from 0.25 ounces of gold and 0.15 ounces of platinum. Suppose that the cost of gold is decreasing at an instantaneous rate of \$20 per ounce per year while the cost of platinum is increasing at an instantaneous rate of \$30 per ounce per year. At what instantaneous rate is the value of Pat's ring increasing or decreasing? *Remember to include units in your answer.*

Answer: The value of Pat's ring is (circle one) INCREASING DECREASING

at a rate of ______________________________

b. [5 points] Jordan wants to design a ring made entirely of gold and platinum with a current value of \$900. Currently, gold costs \$1200 per ounce and platinum costs \$1500 per ounce. Let $w(p)$ be the total weight of Jordan's ring, in ounces, if p ounces of platinum are used.

(i) In the context of this problem, what is the domain of $w(p)$?

Answer: ______________________________

(ii) Find a formula for $w(p)$. No variables other than p should appear in your answer.

Answer: $w(p) =$ ______________________________

(iii) How much gold and platinum should be in the ring if Jordan wants to minimize the weight of the ring? *You do not need to justify your answer.*

Answer: __________ ounces of gold and __________ ounces of platinum

7. [7 points] The record time for the 100 meter dash is 9.58 seconds, set by Usain Bolt at a race in 2009. Let $v(t)$ be Bolt's velocity, in meters per second, t seconds after Bolt starts the race. Several values of $v(t)$ are shown below. Assume that $v(t)$ is an increasing function for the first three seconds of the race.

t	0	0.5	1	1.5	2	2.5	3
$v(t)$	0.6	3.5	5.8	7.7	9.1	10.1	10.6

(Numbers are based on data collected at the 12th IAAF World Championships in Athletics.[1])

a. [2 points] Estimate Bolt's instantaneous acceleration 1.75 seconds after Bolt starts the race. *Remember to include units.*

Answer: ________________________

b. [3 points] Based on the data provided, give the best possible <u>underestimate</u> of the distance run by Bolt during the first 3 seconds of his race. *Be sure to show your work, and remember to include units.*

Answer: ________________________

c. [2 points] How often would we need to measure Bolt's velocity so that the difference between the best possible underestimate and the best possible overestimate of the distance he runs in the first 3 seconds is 2 meters? *Be sure to show your work.*

Answer: Velocity must be measured every ____________________ seconds.

[1] Data available at `http://berlin.iaaf.org`

8. [11 points] A function $g(t)$ and its derivative are given by

$$g(t) = 10e^{-0.5t}(t^2 - 2t + 2) \qquad \text{and} \qquad g'(t) = -10e^{-0.5t}(0.5t^2 - 3t + 3).$$

a. [2 points] Find the t-coordinates of all critical points of $g(t)$. If there are none, write NONE. For full credit, you must find the exact t-coordinates.

Answer: Critical point(s) at $t =$ ____________________

b. [6 points] For each of the following, find the values of t that maximize and minimize $g(t)$ on the given interval. Be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE in the answer blank if appropriate.

(i) Find the values of t that maximize and minimize $g(t)$ on the interval $[0, 8]$.

Answer: Global max(es) at $t =$ __________ Global min(s) at $t =$ __________

(ii) Find the values of t that maximize and minimize $g(t)$ on the interval $[4, \infty)$.

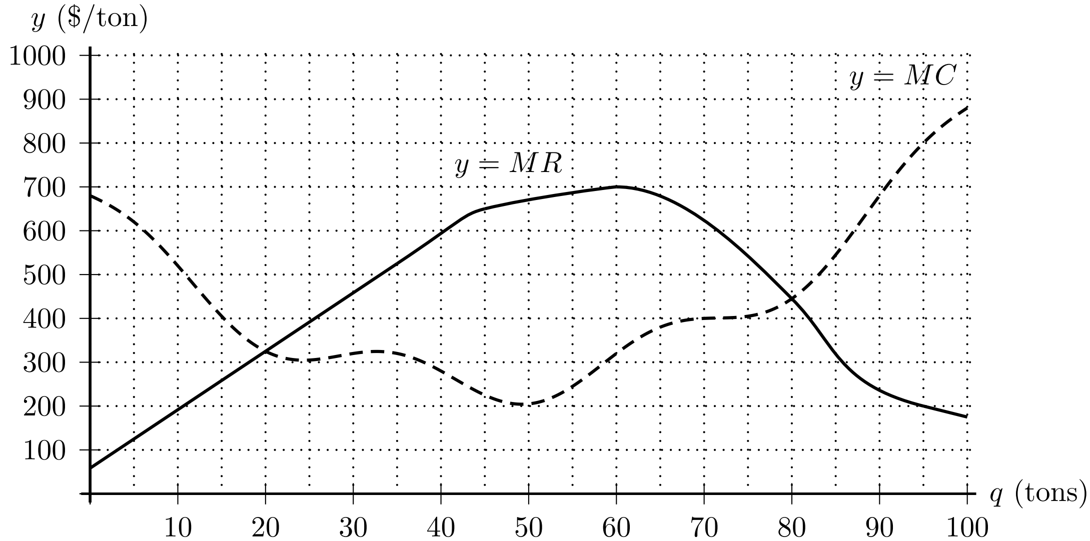
Answer: Global max(es) at $t =$ __________ Global min(s) at $t =$ __________

c. [3 points] Let $G(t)$ be the antiderivative of $g(t)$ with $G(0) = -5$. Find the t-coordinates of all critical points and inflection points of $G(t)$. For each answer blank, write NONE if appropriate. You do not need to justify your answers.

Answer: Critical point(s) at $t =$ ____________________

Answer: Inflection point(s) at $t =$ ____________________

9. [9 points] With winter past and summer approaching, David is opening a business selling ice. Graphed below are his marginal revenue MR (solid line) and marginal cost MC (dashed line), in dollars per ton of ice.



a. [4 points] Carefully estimate the answer to each of the following based on the graphs above. You do not need to show your work.

(i) For what value(s) of q in the interval $[0, 100]$ is revenue maximized?

Answer: $q =$ ____________

(ii) For what value(s) of q in the interval $[0, 100]$ is MR maximized?

Answer: $q =$ ____________

(iii) For what value(s) of q in the interval $[0, 100]$ is profit maximized?

Answer: $q =$ ____________

(iv) For what value(s) of q in the interval $[0, 100]$ is $MR - MC$ maximized?

Answer: $q =$ ____________

b. [2 points] David is planning to sell 5 tons of ice but is considering selling 35 tons instead.

(i) Would David's profit increase or decrease if he changed the amount of ice sold from 5 tons to 35 tons? (Circle one.)

INCREASE DECREASE

(ii) By how much would his profit increase or decrease? (Circle the one best estimate.)

\$1000 \$2000 \$4500 \$5250 \$6000

c. [3 points] Let $\pi(q)$ be David's profit, in dollars, if he sells q tons of ice. Suppose that David would make a profit of \$4000 if he sold 95 tons of ice. Find an equation for the tangent line to the graph of $y = \pi(q)$ at $q = 95$.

Answer: ______________________________

10. [11 points] In each situation, circle all of the statements I-VI which must be true. If none of the statements must be true, circle VII. NONE OF THE ABOVE.

a. [3 points] Let $f(x) = qe^{rx} + s$, where q, r, and s are negative constants.

I. $f(0) > 0$

II. $f'(0) > 0$

III. $\lim\limits_{x \to \infty} f(x) = s$

IV. $\lim\limits_{x \to \infty} f(x) = 0$

V. $\lim\limits_{x \to -\infty} f(x) = s$

VI. $\lim\limits_{x \to -\infty} f(x) = 0$

VII. NONE OF THE ABOVE

b. [4 points] Let $g(x) = a \ln(bx)$, where a and b are positive constants.

I. The domain of $g(x)$ is the interval $(0, \infty)$.

II. The graph of $g(x)$ has a horizontal asymptote.

III. The graph of $g(x)$ has a vertical asymptote.

IV. $g^{-1}(0) = b^{-1}$

V. $g'(x) = \dfrac{a}{bx}$

VI. $\displaystyle\int g(x)\, dx = ax(\ln(bx) - 1) + C$

VII. NONE OF THE ABOVE

c. [4 points] Let $z(t) = A \sin t + B$, where A and B are positive constants.

I. The maximum value of $z(t)$ on its domain is $A + B$.

II. $z(t)$ has an inflection point at $t = 0$.

III. If $h(t) = z(z(t))$, then $h'(0) = A^2 \cos B$.

IV. $\displaystyle\int_0^{2\pi} z(t)\, dt = 0$

V. $\displaystyle\int_0^{\pi} z(t)\, dt = 2A + \pi B$

VI. $\displaystyle\int_1^{2} z(t)\, dt = \int_{1+2\pi}^{2+2\pi} z(t)\, dt$

VII. NONE OF THE ABOVE