Math 115 — First Midterm

February 11, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	12	
3	8	
4	9	
5	13	
6	11	
7	15	
8	8	
9	3	
10	10	
Total	100	

1. [11 points] The table below gives several values of a continuous, invertible function $f(x)$. Assume that the domain of both $f(x)$ and $f'(x)$ is the interval $(-\infty, \infty)$.

x	0	3	6	9	12	15	18	21	24
$f(x)$	-7	-3.5	-2	3	4.5	6	7	9	19

a. [3 points] Evaluate each of the following.

(i) $f(f(15))$

Solution: $f(f(15)) = f(6) = -2.$

Answer: $f(f(15)) =$ -2

(ii) $f^{-1}(3)$

Answer: $f^{-1}(3) =$ 9

(iii) $f^{-1}(2f(12))$

Solution: $f^{-1}(2f(12)) = f^{-1}(2(4.5)) = f^{-1}(9) = 21.$

Answer: $f^{-1}(2f(12)) =$ 21

b. [2 points] Compute the average rate of change of f on the interval $3 \le x \le 18$.

Solution: This average rate of change is equal to the difference quotient

$$\frac{f(18) - f(3)}{18 - 3} = \frac{7 - (-3.5)}{15} = \frac{10.5}{15} = \frac{7}{10} = 0.7.$$

Answer: $10.5/15 = 7/10 = 0.7$

c. [2 points] Estimate $f'(19)$.

Solution: We approximate $f'(19)$ by the average rate of change of f on the interval $18 \le x \le 21$.

$$f'(19) \approx \frac{f(21) - f(18)}{21 - 18} = \frac{9 - 7}{3} = \frac{2}{3}.$$

Answer: $f'(19) \approx$ $2/3 \approx 0.67$

d. [2 points] Let $g(x) = f^{-1}(x)$. Estimate $g'(5)$.

Solution: We approximate $g'(5)$ by the average rate of change of $g(x)$ on the interval $4.5 \le x \le 6$.

$$g'(5) \approx \frac{g(6) - g(4.5)}{6 - 4.5} = \frac{f^{-1}(6) - f^{-1}(4.5)}{1.5} = \frac{15 - 12}{1.5} = \frac{3}{1.5} = 2.$$

Answer: $g'(5) \approx$ $3/1.5 = 2$

e. [2 points] Suppose $f'(0) = 2$. Find an equation for the tangent line to the graph of $y = f(x)$ at $x = 0$.

Solution: This is the line with slope $f'(0) = 2$ that passes through the point $(0, f(0)) = (0, -7)$. An equation for this line is $y = 2x - 7$.

Answer: $y = 2x - 7$

2. [12 points] A scientist is growing a very large quantity of mold. Initially, the mass of mold grows exponentially, but after many hours, the mass stabilizes at 24 kilograms.
Suppose that t hours after the scientist begins, the mass of mold, in kilograms, can be modeled by the function M defined by the equation

$$M(t) = \begin{cases} 0.41e^{0.72t} & \text{if } 0 \le t \le 5 \\ \dfrac{2t^3}{at^b + c} & \text{if } t > 5. \end{cases}$$

a. [4 points] Find the value of k between 0 and 5 so that $M(k) = 1$. Then interpret the equation $M(k) = 1$ in the context of this problem. Use a complete sentence and include units.

Solution: Because we want to find k between 0 and 5, we use the first piece of the formula for M and solve for k in the equation $0.41e^{0.72k} = 1$.

$$\begin{aligned} 0.41e^{0.72k} &= 1 \\ e^{0.72k} &= 1/0.41 \approx 2.439 \\ 0.72k &= \ln(1/0.41) \approx 0.892 \\ k &= \ln(1/0.41)/0.72 \approx 1.238 \end{aligned}$$

Answer: $k =$ $\frac{\ln(1/0.41)}{0.72} \approx 1.238$

Interpretation:

Solution: 1.238 hours after the scientist begins, the mold has a mass of 1 kg.

b. [8 points] Assuming that M is a continuous function of t, determine $\lim_{t\to\infty} M(t)$, and find the values of a, b, and c.

Solution:

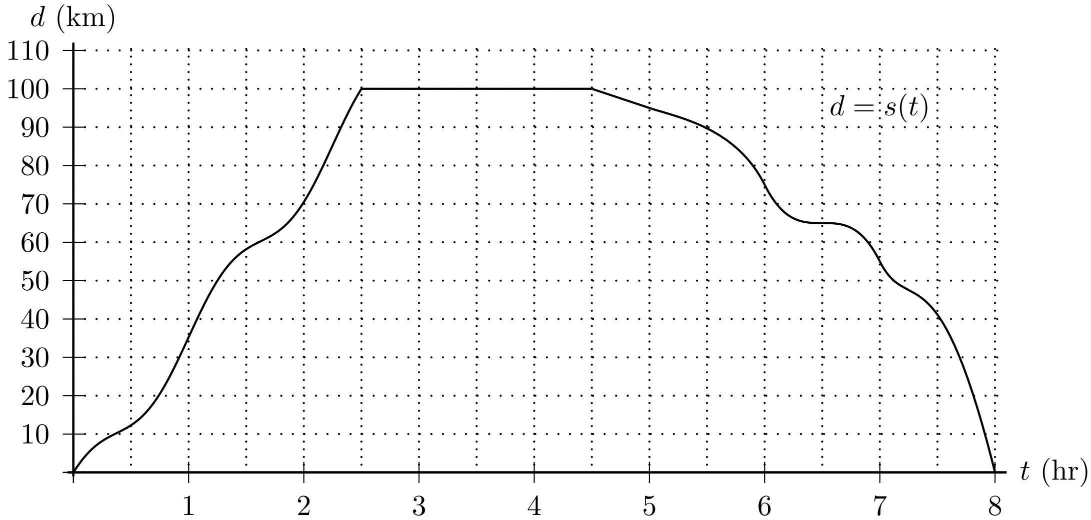
$$\frac{2}{a} = 24 \text{ so } a = 1/12 \approx 0.083$$

$$\begin{aligned} 0.41e^{0.72\cdot 5} &= \frac{2\cdot 5^3}{5^3/12 + c} \\ 0.41e^{3.6} &= \frac{250}{125/12 + c} \\ \frac{125}{12} + c &= \frac{250}{0.41e^{3.6}} \\ c &= \frac{250}{0.41e^{3.6}} - \frac{125}{12} \approx 6.244 \end{aligned}$$

Answers: $\lim_{t\to\infty} M(t) =$ 24 $\quad a =$ $1/12 \approx 0.083$

$b =$ 3 $\quad c =$ $\frac{250}{0.41e^{3.6}} - \frac{125}{12} \approx 6.244$

3. [8 points] A ship's captain is making a round trip voyage between two ports. The ship sets sail from Port Jackson at noon, arrives at Port Kembla some time later, waits there for a while, and then returns to Port Jackson. Let $s(t)$ be the ship's distance, in kilometers, from its starting point of Port Jackson, t hours after noon. A graph of $d = s(t)$ is shown below.



Remember to include units where appropriate.

a. [1 point] How far is Port Kembla from Port Jackson?

Answer: **100 kilometers**

b. [1 point] How long does the ship wait in Port Kembla?

Answer: **2 hours**

c. [1 point] Sometime after 5 PM, there is a time when the ship's instantaneous velocity is 0 km/hr. At what time does this occur?

Answer: **6:30 PM**

d. [2 points] What is the ship's average speed during the return trip from Port Kembla to Port Jackson?

Answer: $100/3.5 \approx 28.6$ **km/hr**

e. [3 points] Estimate the ship's instantaneous velocity at 1 PM.

Answer: 65 **km/hr**

4. [9 points] Let $P(v) = \begin{cases} v^2\sin\left(\frac{1}{v}\right) - v\sin(2) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$

a. [5 points]
Use the limit definition of the derivative to write down an explicit expression for $P'(0)$. Your answer should not include the letter P.
Do not attempt to evaluate or simplify the limit.

$$P'(0) = \lim_{h\to 0} \frac{\left((0+h)^2\sin\left(\frac{1}{0+h}\right) - (0+h)\sin(2)\right) - 0}{h}$$

b. [4 points] Use your answer to (a) to estimate $P'(0)$ to the nearest hundredth.
Be sure to include enough clear graphical or numerical evidence to justify your answer.

Solution: We plug in small values of h approaching 0. Since the difference quotient is an even function of h, we need only check positive values of h (as evenness implies that negative h give precisely the same results).

$h = 0.1$:
$$\frac{0.1^2\sin(1/0.1) - 0.1\sin(2) - 0}{0.1} \approx -0.964$$

$h = 0.01$:
$$\frac{0.01^2\sin(1/0.01) - 0.01\sin(2) - 0}{0.01} \approx -0.914$$

$h = 0.001$:
$$\frac{0.001^2\sin(1/0.001) - 0.001\sin(2) - 0}{0.001} \approx -0.908$$

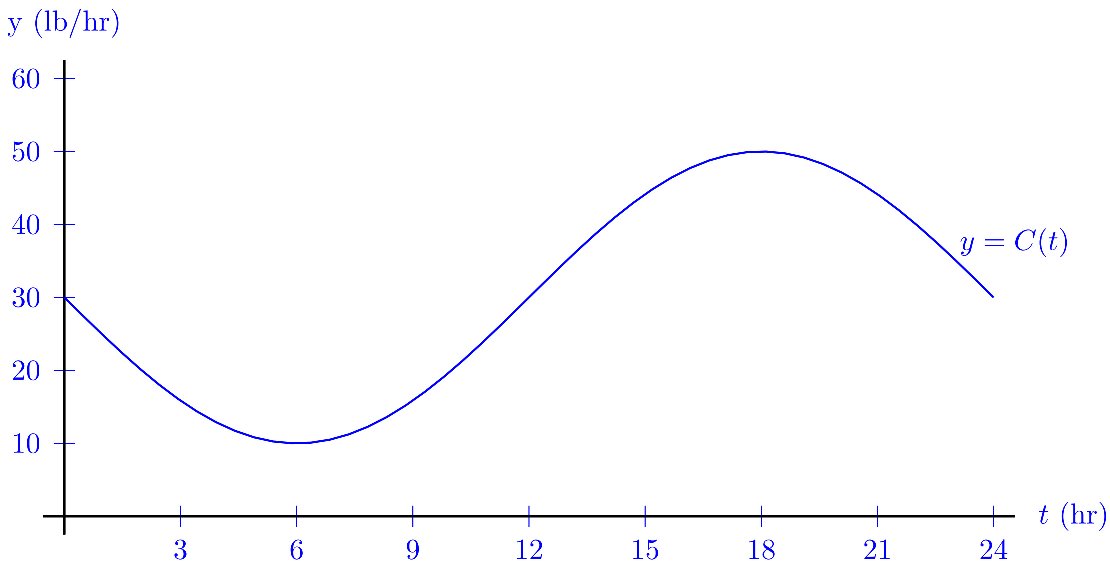
$h = 0.0001$:
$$\frac{0.0001^2\sin(1/0.0001) - 0.0001\sin(2) - 0}{0.0001} \approx -0.909$$

We see at this point that the numbers seem to have stabilized to the nearest hundredth at -0.91.

Answer: $P'(0) \approx$ -0.91

5. [13 points] Jordan owns a 24-hour coffee shop. The coffee brewing rate (or CBR) at Jordan's coffee shop varies throughout the day. The CBR is highest at 6 AM, when coffee is brewed at a rate of 50 pounds of coffee per hour. It is lowest at 6 PM, when coffee is brewed at a rate of only 10 pounds of coffee per hour. Suppose that t hours after noon, the CBR, in pounds of coffee per hour, of Jordan's coffee shop can be modeled by a sinusoidal function $C(t)$ with period 24 hours.

a. [4 points] On the axes provided below, sketch a well-labeled graph of $C(t)$ for $0 \le t \le 24$.



b. [4 points] Find a formula for $C(t)$.

Answer: $C(t) =$ $-20\sin\left(\frac{\pi}{12}t\right) + 30$

c. [5 points] For how many hours each day is the CBR of Jordan's shop at least 40 pounds of coffee per hour? *Remember to show your work.*

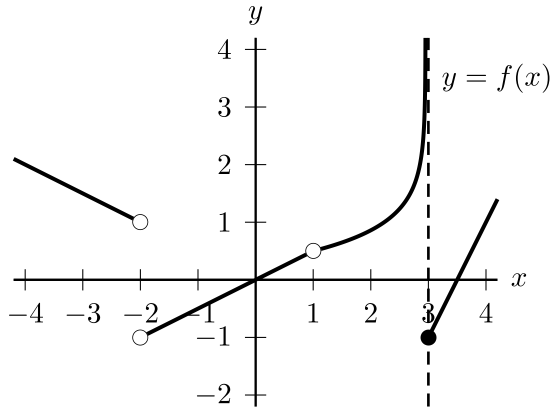
Solution: We wish to find the two solutions to $C(t) = 40$ for $0 \le t \le 24$. We start by finding any solution:

$$\begin{aligned} -20\sin\left(\frac{\pi}{12}t\right) + 30 &= 40 \\ \sin\left(\frac{\pi}{12}t\right) &= -0.5 \\ \frac{\pi}{12}t &= \arcsin(-0.5) = -\pi/6 \\ t &= -2. \end{aligned}$$

One of the solutions we want is therefore $t = -2 + 24 = 22$, and by symmetry around the peak at 18, the other is $t = 14$.
Therefore, the CBR is at least 40 for the 8 hours between $t = 14$ and $t = 22$.

Answer: **8 hours**

6. [11 points] Below is the graph of a portion of a function $f(x)$.



a. [2 points] Give all values of a in the interval $-4 < a < 4$ that are not in the domain of $f(x)$. If there are none, write NONE.

Answer: -2, 1

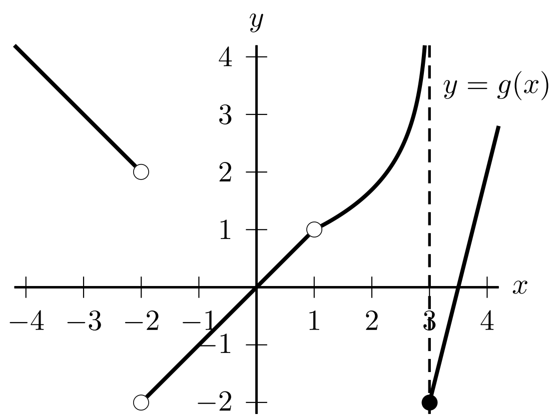
b. [2 points] Give all values of a in the interval $-4 < a < 4$ where $f(x)$ is not continuous at $x = a$. If there are none, write NONE.

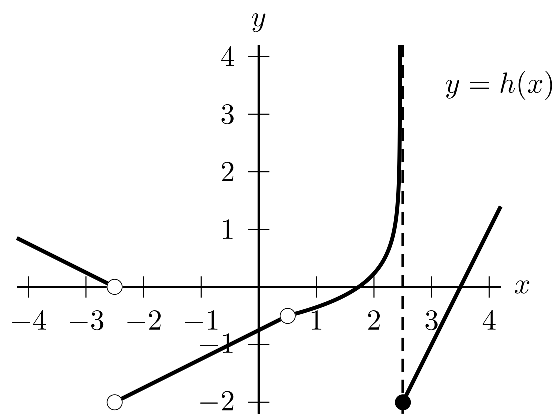
Answer: -2, 1, 3

c. [2 points] Give all values of a in the interval $-4 < a < 4$ where $\lim_{x \to a} f(x)$ does not exist. If there are none, write NONE.

Answer: -2, 3

d. [5 points] The graphs below show portions of two other functions $g(x)$ and $h(x)$ which are transformations of $f(x)$. Express $g(x)$ and $h(x)$ as transformations of $f(x)$.





Answer: $g(x) =$ $2f(x)$ and $h(x) =$ $f(x + 0.5) - 1$

7. [15 points] During the winter, the town of Waterville uses salt to keep the roads from freezing. Let $S = f(T)$ be the amount of salt, in tons, used on the roads of Waterville on a day when the average temperature is T °F. Let $C = g(S)$ be the cost, in thousands of dollars, of S tons of salt. Assume that both f and g are invertible functions that are differentiable everywhere.

a. [3 points] Interpret the equation $f^{-1}(4) = 9$ in the context of this problem.
Use a complete sentence and include units.

Solution: On a day when Waterville uses 4 tons of salt on the roads, the average temperature is 9 °F.

b. [3 points] Interpret the equation $g(f(7)) = 2$ in the context of this problem.
Use a complete sentence and include units.

Solution: On a day when the average temperature is 7 °F, the salt used on Waterville's roads costs \$2000.

c. [2 points] Yesterday, the average temperature in Waterville was w °F.
Give a single mathematical expression equal to the average temperature, in °F, on a day when Waterville uses twice as much salt on the roads as it did yesterday.

Answer: $f^{-1}(2f(w))$

d. [4 points] Give a single mathematical equality involving the derivative of f which supports the following claim:
On a day when the average temperature is 3°F, Waterville uses approximately 0.12 tons less salt on the roads than on a day when the average temperature is 1°F.

Answer: $f'(1) = -0.06$

e. [3 points] In the equation $(g^{-1})'(8) = 5$, what are the units on 8 and 5?

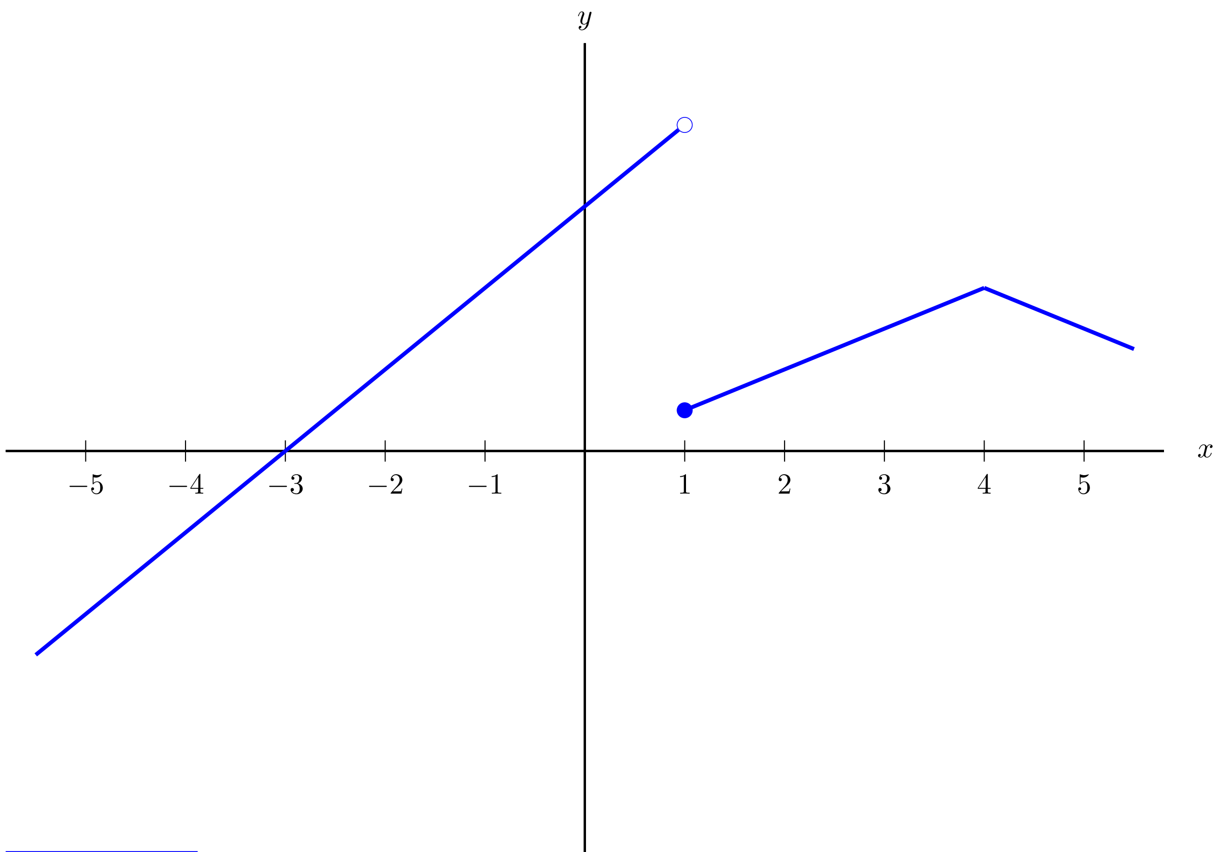
Answer: Units on 8 are **thousands of dollars**

Answer: Units on 5 are **tons of salt per thousand dollars**

8. [8 points] On the axes provided below, sketch the graph of a single function $y = g(x)$ satisfying all of the following:

- $g(x)$ is defined for all x in the interval $-5 < x < 5$.
- $g'(x) > 0$ for all $x < 0$.
- $g(x)$ has a point of discontinuity at $x = 1$.
- The average rate of change of $g(x)$ between $x = -2$ and $x = 2$ is 0.
- $g(x) > 0$ for all $x > 3$.
- $g'(x) < 0$ for all $x > 4$.

Make sure that your sketch is large and unambiguous.



Solution: Many possibilities exist. Note that in order to satisfy the fourth property, we must have $g(-2) = g(2)$.

9. [3 points] Find all vertical and horizontal asymptotes of the graph of

$$g(x) = \frac{k(x-a)(x-b)}{(x-a)(x-c)^2}$$

where a, b, c, and k are constants with $a < b < c < k$. If there are none, write NONE.

Horizontal asymptote(s): $y = 0$

Vertical asymptote(s): $x = c$

10. [10 points] *Throughout this page, give all answers in exact form. Do not use decimal approximations. For example,* $x = \frac{1}{3}$ *is an exact solution to* $3x = 1$*, but* $x = 0.3333333333$ *is not.*
Kathy is making hot chocolate one morning while out camping in the cold. She heats it for ten minutes, during which time its temperature increases at a constant rate from 2°C to 80°C. Let $H(t)$ be the temperature, in °C, of the chocolate, t minutes after Kathy begins heating it.

a. [2 points] Find a formula for $H(t)$ which is valid for $0 < t < 10$.

Solution: The temperature increases at a constant rate, so $H(t)$ is linear with slope equal to the constant average rate of change $\frac{H(10)-H(0)}{10-0} = \frac{80-2}{10} = 7.8$°C/min. Since $H(0) = 2$, we see that $H(t) = 2 + 7.8t$.

Answer: $H(t) =$ $2 + 7.8t$

b. [5 points] After ten minutes, when the chocolate is 80°C, Kathy turns off her camping stove. The temperature of the chocolate begins to decay exponentially so that its temperature, in °C, decreases by 25% every two minutes.
Find a formula for $H(t)$ which is valid for $t \geq 10$.

Solution: Since the temperature decays exponentially, for $t \geq 10$, there are constants c and a so that $H(t) = ca^t$. Because the temperature decreases by 25% every two minutes, $a^2 = 0.75$. (To see this, note that since $H(10) = 80$ and $H(12) = 0.75(80) = 60$, we have $ca^{10} = 80$ and $ca^{12} = 60$. Taking the ratios of the two sides of these equations we find $a^2 = \frac{60}{80} = 0.75$.)
Thus $a = (0.75)^{1/2}$ and we solve for c in the equation $80 = c(0.75)^{(1/2)(10)}$ to find that $c = \frac{80}{(0.75)^5}$. Hence, for $t \geq 10$, we have $H(t) = \frac{80}{(0.75)^5}(0.75)^{0.5t}$. (Note this can also be written as $H(t) = 80(0.75)^{(t-10)/2}$.)

Answer: $H(t) =$ $\dfrac{80}{(0.75)^5}(0.75)^{0.5t}$ **or** $80(0.75)^{(t-10)/2}$

When Kathy gets home, she discovers that her water bottle is full of ice. From the moment she gets home, it takes 30 minutes for the ice to melt completely. Let $V(t)$ be the volume, in cubic inches, of the ice in Kathy's water bottle, t minutes after she gets home. Until the ice is gone, a formula for V is given by the equation $V(t) = -4\ln(kt+b)$ for some constants k and b.

c. [3 points] Find the value of k in terms of b.

Solution: Since $V(30) = 0$, we solve for k in the equation $-4\ln(k(30) + b = 0$.

$$-4\ln(k(30) + b) = 0$$
$$\ln(30k + b) = 0$$
$$30k + b = 1 (= e^0)$$
$$k = \frac{1-b}{30}$$

Answer: $k =$ $\dfrac{1-b}{30}$