Math 115 — Second Midterm

March 22, 2016

EXAM SOLUTIONS

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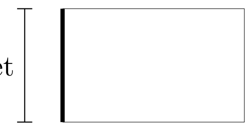
1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your initials (not name) on every page and point this out to your instructor when you hand in the exam.
4. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. The use of any networked device while working on this exam is <u>not</u> permitted.
8. You may use any calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
9. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
10. Include units in your answer where that is appropriate.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	7	
2	12	
3	9	
4	10	
5	15	

Problem	Points	Score
6	11	
7	7	
8	14	
9	6	
10	9	
Total	100	

1. [7 points] Gertrude wants to enclose a rectangular region in her backyard. She wants to use high fencing (thick line), which costs \$200 per foot, for one side of the rectangle. For the remaining three sides, she wants to use normal fencing (thin line), which costs \$75 per foot. Let $A(h)$ be the area (in square feet) of the region enclosed by the fence if h is the length (in feet) of the side with high fencing and Gertrude spends \$3000 on fencing for the project.



a. [4 points] Find a formula for $A(h)$.

Solution: Let ℓ be the other sidelength of the rectangle. Then, the total cost of the fencing is

$$200h + 75(2\ell + h) = 275h + 150\ell.$$

If the total cost of fencing is \$3000, then

$$\begin{aligned} 275h + 150\ell &= 3000 \\ 150\ell &= 3000 - 275h \\ \ell &= 20 - \frac{11}{6}h. \end{aligned}$$

Hence,

$$A(h) = h\ell = 20h - \frac{11}{6}h^2.$$

Answer: $A(h) =$ $20h - \dfrac{11}{6}h^2$

b. [3 points] In the context of this problem, what is the domain of $A(h)$?

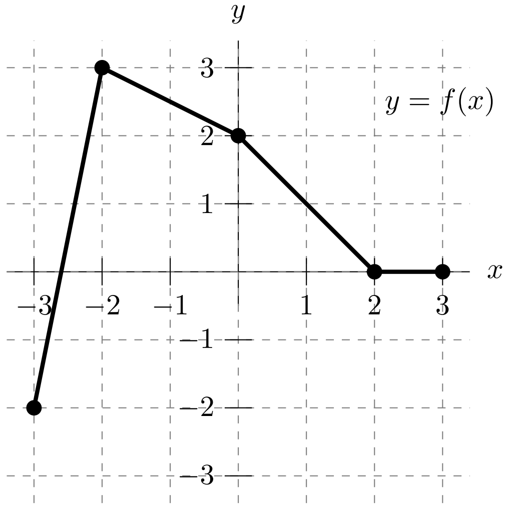
Solution: Note that $h > 0$, or else we would not have a rectangle. Note also that $\ell > 0$ (where ℓ is the other sidelength).
So since $275h + 150\ell = 3000$, we have $275h = 3000 - 150\ell < 3000$, so $h < \frac{3000}{275} = \frac{120}{11} \approx 10.91$. Hence, the domain of $A(h)$ is $0 < h < \frac{120}{11}$.

(Note that in this situation, it would also be okay to include the endpoints 0 and 3000/275, which correspond to the degenerate cases of a rectangle of length or width 0.)

Answer: Domain: **The interval** $\left(0, \dfrac{120}{11}\right)$ **(or** $[0, \frac{120}{11}]$**)**

2. [12 points]

Let f be the piecewise linear function with graph shown below.



The table below gives several values of a differentiable function g and its derivative g'.
Assume that both $g(x)$ and $g'(x)$ are invertible.

x	-2	-1	0	2	5
$g(x)$	21	11	5	-1	-3
$g'(x)$	-12	-8	-4	-2	-0.4

You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

For each of parts **a.-f.** below, find the value of the given quantity. If there is not enough information provided to find the value, write "NOT ENOUGH INFO". If the value does not exist, write "DOES NOT EXIST".

a. [2 points] Let $j(x) = e^{g(x)}$. Find $j'(2)$.

Answer: $-\dfrac{2}{e} \approx -0.736$

b. [2 points] Let $k(x) = f(x)f(x+2)$. Find $k'(-1)$.

Answer: -3

c. [2 points] Let $h(x) = 3f(x) + g(x)$. Find $h'(-2)$.

Answer: **DOES NOT EXIST**

d. [2 points] Find $(g^{-1})'(2)$.

Answer: **NOT ENOUGH INFO**

e. [2 points] Let $m(x) = g(f(g(x)))$. Find $m'(2)$.

Answer: **NOT ENOUGH INFO**

f. [2 points] Let $\ell(x) = \dfrac{f(x)}{g(2x)}$. Find $\ell'(-1)$.

Answer: $\dfrac{21(-.5) - 2(2.5)(-12)}{21^2} \approx 0.1122$

3. [9 points] Consider the curve $\mathcal{C}$ defined by

$$\cos(ax - y) + x^2 + y^2 = b$$

where a and b are positive constants.

a. [5 points] For the curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution:
Implicit differentiation:

$$\frac{d}{dx}\left(\cos(ax-y) + x^2 + y^2\right) = \frac{d}{dx}(b)$$
$$\left(a - \frac{dy}{dx}\right)(-\sin(ax-y)) + 2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$-a\sin(ax-y) + \frac{dy}{dx}(\sin(ax-y)) + 2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(\sin(ax-y) + 2y) = a\sin(ax-y) - 2x$$
$$\frac{dy}{dx} = \frac{a\sin(ax-y) - 2x}{\sin(ax-y) + 2y}$$

Answer: $\frac{dy}{dx} =$ $\frac{a\sin(ax-y) - 2x}{\sin(ax-y) + 2y}$

b. [1 point] Let $a = 1$ and $b = 9$. Exactly one of the following points (x, y) lies on the curve $\mathcal{C}$. Circle that one point.

$(3, 0)$ $\boxed{(2, 2)}$ $(1, -1)$ (π, π) $(0, -9)$

c. [3 points] With $a = 1$ and $b = 9$ as above, find an equation for the tangent line to the curve $\mathcal{C}$ at the point you chose in part **b.**.

Solution: To find the slope of the tangent line, we evaluate $\frac{dy}{dx}$ at the point $(2, 2)$ to find

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,2)} = \frac{\sin(2-2) - 2(2)}{\sin(2-2) + 2(2)} = -1.$$

Hence, the equation for the tangent line is $y = 2 - 1(x - 2) = -x + 4$.

Answer: $y =$ $2 - (x - 2)$ **(or** $-x + 4$**)**

4. [10 points] Let $h(x)$ be a twice differentiable function defined for all real numbers x. (So h is differentiable and its derivative h' is also differentiable.)
Some values of $h'(x)$, the derivative of h are given in the table below.

x	-8	-6	-4	-2	0	2	4	6	8
$h'(x)$	3	7	0	-3	-5	-4	0	-2	6

For each of the following, circle all the correct answers.
Circle "NONE OF THESE" if none of the provided choices are correct.

a. [2 points] Circle all the intervals below in which $h(x)$ must have a critical point.

$-8 < x < -6$ $\boxed{-6 < x < -2}$ $-2 < x < 2$ $\boxed{2 < x < 6}$ $\boxed{6 < x < 8}$

NONE OF THESE

b. [2 points] Circle all the intervals below in which $h(x)$ must have a local extremum (i.e. a local maximum or a local minimum).

$-8 < x < -6$ $\boxed{-6 < x < -2}$ $-2 < x < 2$ $2 < x < 6$ $\boxed{6 < x < 8}$

NONE OF THESE

c. [2 points] Circle all the intervals below in which $h(x)$ must have an inflection point.

$\boxed{-8 < x < -4}$ $-4 < x < 0$ $0 < x < 4$ $\boxed{2 < x < 6}$ $\boxed{4 < x < 8}$

NONE OF THESE

d. [2 points] Circle all the intervals below which must contain a number c such that $h''(c) = 2$.

$\boxed{-8 < x < -6}$ $-4 < x < -2$ $-2 < x < 0$ $\boxed{2 < x < 4}$ $6 < x < 8$

NONE OF THESE

e. [2 points] Suppose that $h''(x) < 0$ for $x < -8$, and $h(-8) = 7$. Circle all the numbers below which could equal the value of $h(-10)$.

$\boxed{-2}$ $\boxed{-1}$ $\boxed{0}$ 1 2

NONE OF THESE

5. [15 points] Suppose $g(x)$ is a differentiable function defined for all real numbers x. The <u>derivative</u> and <u>second derivative</u> of $g(x)$ are given by

$$g'(x) = x^2(x+4)(x+2)^{1/3} \qquad \text{and} \qquad g''(x) = \frac{2x(x+3)(5x+8)}{3(x+2)^{2/3}}.$$

a. [2 points] Find the x-coordinates of all critical points of $g(x)$.
If there are none, write "NONE".

Answer: Critical point(s) of $g(x)$ at $x =$ $-4, -2, 0$

b. [2 points] Find the x-coordinates of all critical points of $g'(x)$.
If there are none, write "NONE".

Answer: Critical point(s) of $g'(x)$ at $x =$ $-3, -2, -\frac{8}{5}, 0$

c. [6 points] Find the x-coordinates of all local maxima and local minima of $g(x)$.
If there are none of a particular type, write "NONE". Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

Solution: We know from part **a.** that the critical points of $g(x)$ are $-4, -2, 0$. Notice that the Second Derivative Test is inconclusive, because $g''(-2)$ does not exist and $g''(0) = 0$. So we must use the First Derivative Test. Notice that the factor x^2 is always non-negative, $(x+4)$ is negative for $x < -4$ and positive for $x > -4$, and $(x+2)^{1/3}$ is negative for $x < -2$ and positive for $x > -2$. This gives us the resulting signs:

Interval	$x < -4$	$-4 < x < -2$	$-2 < x < 0$	$x > 0$
Sign of $g'(x)$	$+ \cdot - \cdot - = +$	$+ \cdot + \cdot - = -$	$+ \cdot + \cdot + = +$	$+ \cdot + \cdot + = +$

So $g(x)$ has a local maximum at $x = -4$ and a local minimum at $x = -2$. There is neither a local maximum nor a local minimum at $x = 0$.

Answer: Local max(es) at $x =$ -4 and Local min(s) at $x =$ -2

d. [5 points] Find the x-coordinates of all inflection points of $f(x)$.
If there are none, write "NONE". Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: We know from part **b.** that $g''(x)$ is 0 or undefined (i.e., $g'(x)$ has a critical point or is undefined) at $x = -3, -2, -\frac{8}{5}, 0$. To determine whether these are actually inflection points (where concavity changes), we must test the sign of the second derivative on either side of each point. Notice that the factor $2x$ is negative for $x < 0$ and positive for $x > 0$, the factor $(x+3)$ is negative for $x < -3$ and positive for $x > -3$, the factor $(5x+8)$ is negative for $x < -\frac{8}{5}$ and positive for $x > -\frac{8}{5}$, and the factor $(x+2)^{-2/3}$ is always non-negative. We find the following signs for $g''(x)$:

Interval	$x < -3$	$-3 < x < -2$	$-2 < x < -\frac{8}{5}$	$-\frac{8}{5} < x < 0$	$x > 0$
Sign of $g''(x)$	$\frac{-\cdot-\cdot-}{+} = -$	$\frac{-\cdot+\cdot-}{+} = +$	$\frac{-\cdot+\cdot-}{+} = +$	$\frac{-\cdot+\cdot+}{+} = -$	$\frac{+\cdot+\cdot+}{+} = +$

So $g(x)$ has inflection points at $x = -3$, $x = -\frac{8}{5}$, and $x = 0$ but it does <u>not</u> have an inflection point at $x = -2$.

Answer: Inflection point(s) at $x =$ $-3, -\frac{8}{5}, 0$

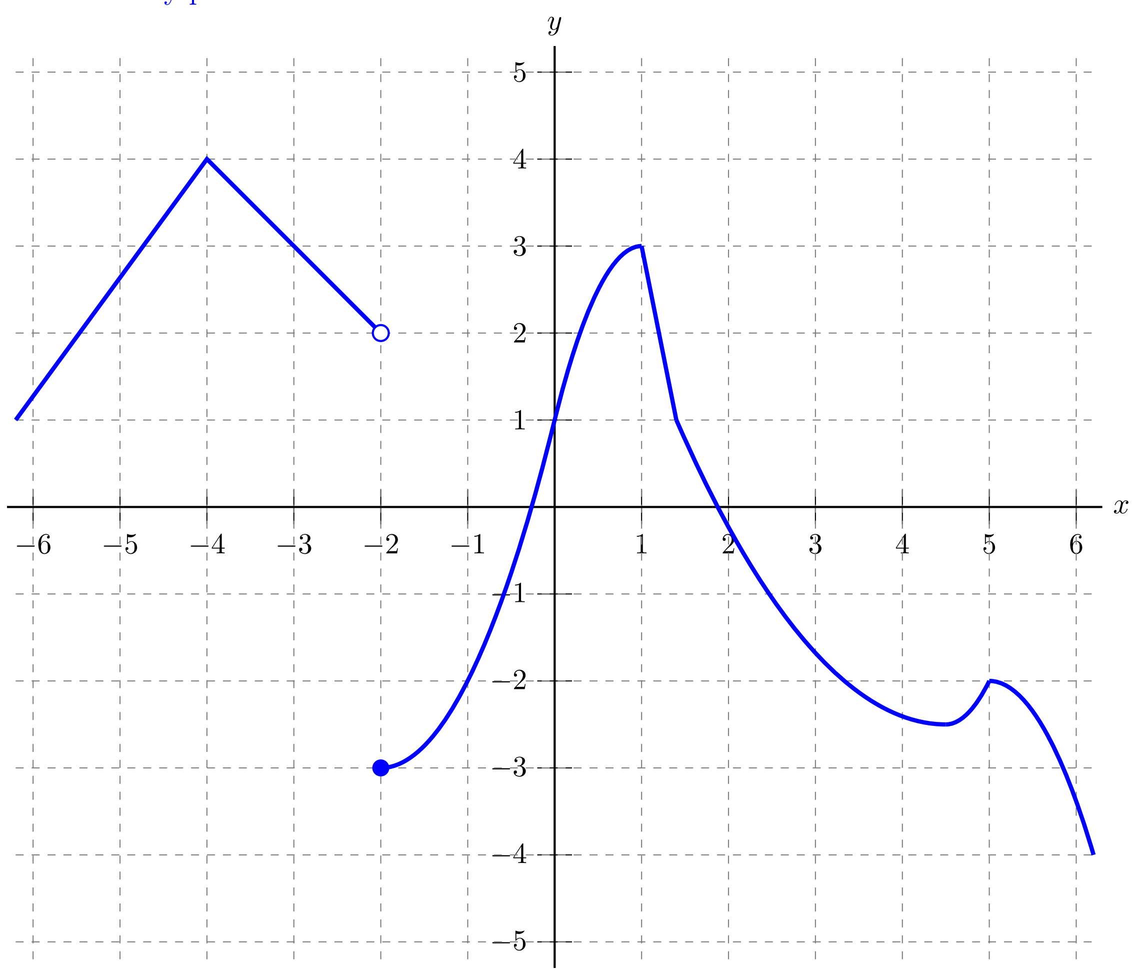
6. [11 points] On the axes provided below, sketch the graph of a single <u>function</u> $y = g(x)$ satisfying all of the following:

- $g(x)$ is defined for <u>all</u> x in the interval $-6 < x < 6$.
- $g(x)$ has at least <u>5</u> critical points in the interval $-6 < x < 6$.
- The global maximum value of $g(x)$ on the interval $-5 \leq x \leq -3$ is 4, and this occurs at $x = -4$.
- $g(x)$ is <u>not</u> continuous at $x = -2$.
- $g'(x)$ (the derivative of g) has a local maximum at $x = 0$.
- $g(x)$ is continuous but <u>not</u> differentiable at $x = 1$.
- $g''(x) \geq 0$ for all x in the interval $2 < x < 4$.
- $g(x)$ has at least one local minimum on the interval $4 < x < 6$ but does not have a global minimum on the interval $4 < x < 6$.
- $g(x)$ has an inflection point at $x = 5$.

Make sure your sketch is large and unambiguous.

$\boxed{\text{Graph of } y = g(x)}$

There are many possible solutions. One is shown below.



7. [7 points] Alicia decides to go for a run before completing her math homework. Let $g(m)$ be the time (in hours) that Alicia spends completing her math assignment if she runs m miles. Suppose that for $1.2 \le m \le 8$,

$$g(m) = 2m - 12.2\ln(m) + 15 - \frac{14.4}{m}.$$

Note that on this interval, the derivative of g is given by the formula

$$g'(m) = \frac{2(m-4.5)(m-1.6)}{m^2}.$$

a. [5 points] Find all values of m that maximize and minimize the function $g(m)$ on the interval $1.2 \le m \le 8$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

Solution: Since g is continuous on the closed interval $[1.2, 8]$, by the Extreme Value Theorem g definitely attains a global maximum and global minimum on the interval, and it suffices to compare the values of $g(m)$ at the critical points and endpoints of the interval.

Notice that the critical points of $g(m)$ in the interval $1.2 \le m \le 8$ are at $m = 1.6$ and $m = 4.5$. Hence, we need to check the value of $g(m)$ at $m = 1.2, 1.6, 4.5, 8$:

$$g(1.2) = 2(1.2) - 12.2\ln(1.2) + 15 - \frac{14.4}{1.2} \approx 3.176$$
$$g(1.6) = 2(1.6) - 12.2\ln(1.6) + 15 - \frac{14.4}{1.6} \approx 3.466$$
$$g(4.5) = 2(4.5) - 12.2\ln(4.5) + 15 - \frac{14.4}{4.5} \approx 2.450$$
$$g(8) = 2(8) - 12.2\ln(8) + 15 - \frac{14.4}{8} \approx 3.831$$

Thus, we can see that $g(m)$ achieves its maximum on the interval at $m = 8$, and $g(m)$ achieves its minimum on the interval at $m = 4.5$.

For each answer blank below, write "NONE" if appropriate.

Answer: Global max(es) at $m=$ 8

Answer: Global min(s) at $m=$ 4.5

b. [2 points] Assuming that Alicia runs at least 1.2 miles and at most 8 miles, what is the shortest amount of time Alicia could spend completing her homework?
Remember to include units.

Solution: As we saw in part **a.**, under those assumptions, the shortest amount of time Alicia could spend completing her homework is $g(4.5) \approx 2.450$ hours.

Answer: Shortest time: 2.450 **hours**

8. [14 points]
Suppose H is a differentiable function such that $H'(w)$ is also differentiable for $0 < w < 10$. Several values of $H(w)$ and of its first and second derivatives are given in the table on the right.

w	1	2	3	5	8
$H(w)$	6.3	5.4	5.2	4.8	0.7
$H'(w)$	-1.5	-0.4	-0.1	-0.6	-2.1
$H''(w)$	1.6	0.9	0	-0.8	-0.4

Assume that between each pair of consecutive values of w shown in the table, each of $H'(w)$ and $H''(w)$ is either always strictly decreasing or always strictly increasing. *Remember to show your work carefully.*

a. [3 points] Use an appropriate linear approximation to estimate $H(5.2)$.

Solution: For w near 5, local linearization gives $H(w) \approx H(5) + H'(5)(w-5)$, so

$$H(5.2) \approx H(5) + H'(5)(5.2-5) = 4.8 - 0.6(0.2) = 4.8 - 0.12 = 4.68.$$

Answer: $H(5.2) \approx$ 4.68

b. [5 points] Let $J(w)$ be the local linearization of H near $w = 2$, and let $K(w)$ be the local linearization of H near $w = 3$. Which of the following statements must be true? Circle all of the statements that must be true, or circle "NONE OF THESE".

$J(2) > H(2)$	$J(2.5) > H(2.5)$	$\boxed{K(3.5) > H(3.5)}$
$\boxed{J(2) = H(2)}$	$J(2.5) = H(2.5)$	$K(3.5) = H(3.5)$
$J(2) < H(2)$	$\boxed{J(2.5) < H(2.5)}$	$K(3.5) < H(3.5)$
$J'(2) > H'(2)$	$K(2.5) > H(2.5)$	$\boxed{K'(3.5) > H'(3.5)}$
$\boxed{J'(2) = H'(2)}$	$K(2.5) = H(2.5)$	$K'(3.5) = H'(3.5)$
$J'(2) < H'(2)$	$\boxed{K(2.5) < H(2.5)}$	$K'(3.5) < H'(3.5)$

NONE OF THESE

c. [3 points] Use the quadratic approximation of $H(w)$ at $w = 1$ to estimate $H(0.9)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.)

Solution: Let $Q(w)$ be the quadratic approximation of $H(w)$ at $w = 1$.
Then $Q(w) = H(1) + H'(1)(w-1) + \frac{H''(1)}{2}(w-1)^2 = 6.3 - 1.5(w-1) + \frac{1.6}{2}(w-1)^2$.
So, $H(0.9) \approx Q(0.9) = 6.3 - 1.5(0.9-1) + \frac{1.6}{2}(0.9-1)^2 = 6.3 + 0.15 + 0.008 = 6.458$.

Answer: $H(0.9) \approx$ 6.458

d. [3 points] Consider the function N defined by $N(w) = H(2w^2 - 10)$, and let $L(w)$ be the local linearization of $N(w)$ at $w = 3$. Find a formula for $L(w)$. Your answer should not include the function names N or H.

Solution: We know that $L(w) = N(3) + N'(3)(w-3)$.
Note that $N(3) = H(2(3^2) - 10) = H(8) = 0.7$.
To find $N'(3)$, we apply the Chain Rule. In particular, $N'(w) = (4w) \cdot H'(2w^2 - 10)$, so $N'(3) = (4 \cdot 3) \cdot H'(2(3^2) - 10) = 12H'(8) = 12(-2.1) = -25.2$.
Therefore, $L(w) = N(3) + N'(3)(w-3) = 0.7 - 25.2(w-3)$.

Answer: $L(w) =$ $0.7 - 25.2(w-3)$

9. [6 points] Consider a continuous function T with the following properties.

- $T(v)$ is defined for all real numbers v.
- The critical points of $T(v)$ are the four points $v = 3$, $v = 5$, $v = 7$, and $v = 8$. ($T(v)$ has no other critical points.)

Some values of T are shown in the following table:

v	0	3	5	7	8	10
$T(v)$	21	9	13	19	11	21

For each of **a.-f.** below, use the answer blank provided to list all the values v at which $T(v)$ attains the specified global extremum. If there is not enough information provided to give an answer, write "NOT ENOUGH INFO". If $T(v)$ does not attain the specified global extremum on the specified interval, write "NONE".

For what value(s) v does $T(v)$ attain its ...

a. global minimum on the interval $0 \le v \le 10$?

Answer: $v =$ 3

b. global maximum on the interval $0 \le v \le 10$?

Answer: $v =$ 0, 10

c. global minimum on the interval $0 < v < 10$?

Answer: $v =$ 3

d. global maximum on the interval $0 < v < 10$?

Answer: $v =$ NONE

e. global minimum on the interval $(-\infty, \infty)$?

Answer: $v =$ 3

f. global maximum on the interval $(-\infty, \infty)$?

Answer: $v =$ NONE

10. [9 points] Consider the function h defined by $h(x) = \begin{cases} Ax^4 & \text{if } x < 2 \\ Bx^3 + 80\ln\left(\frac{x}{2}\right) & \text{if } x \geq 2 \end{cases}$

where A and B are constants.

a. [6 points] Find values of A and B so that h is differentiable.
Remember to show your work clearly.

Solution: If h is differentiable, it must be continuous, so, in particular,

$$\begin{aligned} \lim_{x\to 2^-} h(x) &= \lim_{x\to 2^+} h(x) \\ A(2)^4 &= B(2)^3 + 80\ln(2/2) \\ 16A &= 8B \\ 2A &= B. \end{aligned}$$

Note that $\frac{d}{dx}\left(Ax^4\right) = 4Ax^3$ and $\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right) = 3Bx^2 + 80(\frac{2}{x})(\frac{1}{2}) = 3Bx^2 + \frac{80}{x}$. and that both Ax^4 and $Bx^3 + 80\ln\left(\frac{x}{2}\right))$ are differentiable at $x = 2$.
In order for $h(x)$ to be differentiable at $x = 2$, $h'(x)$ must exist at $x = 2$. In particular,

$$\begin{aligned} \lim_{k\to 0^-} \frac{h(2+k) - h(2)}{k} &= \lim_{k\to 0^+} \frac{h(2+k) - h(2)}{k} \\ \left(\frac{d}{dx}(Ax^4)\right)\bigg|_{x=2} &= \left(\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right)\right)\bigg|_{x=2} \\ (4Ax^3)\Big|_{x=2} &= \left(3Bx^2 + \frac{80}{x}\right)\bigg|_{x=2} \quad \text{(i.e. derivatives of the two pieces are equal at } x = 2) \\ 32A &= 12B + 40. \end{aligned}$$

Since $B = 2A$, we therefore find that

$$\begin{aligned} 32A &= 24A + 40 \\ 8A &= 40 \\ A &= 5 \end{aligned}$$

and hence $B = 2A = 2(5) = 10$.

Answer: $A =$ 5 and $B =$ 10

b. [3 points] Using the values of A and B you found in part **a.**, find the tangent line approximation for $h(x)$ near $x = 1$.

Solution: First, notice that

$$h(1) = 5(1)^4 = 5$$

and

$$h'(1) = 4(5)(1)^3 = 20.$$

So the tangent line approximation for $h(x)$ near $x = 1$ is $y = 5 + 20(x - 1) = 20x - 15$.

Answer: The tangent line approximation is given by $y =$ $5 + 20(x - 1)$ **(or** $20x - 15$**)**