Math 115 — First Midterm — February 8, 2017

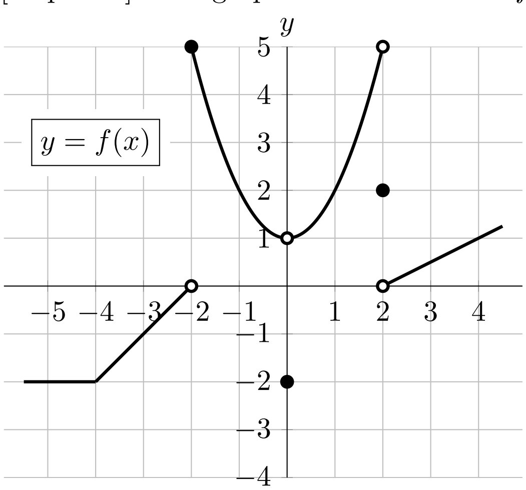
EXAM SOLUTIONS

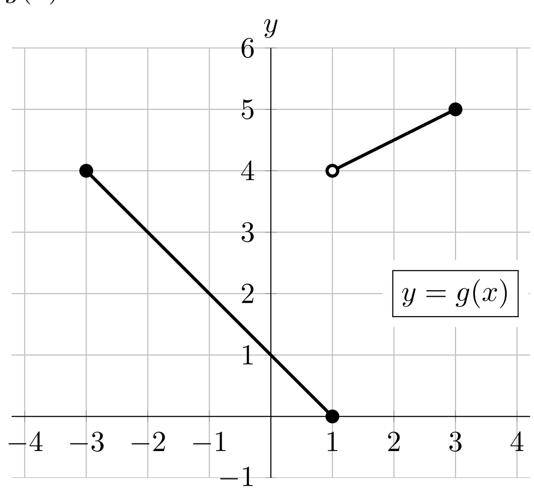
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	19	
2	5	
3	10	
4	10	
5	7	
6	11	

Problem	Points	Score
7	10	
8	11	
9	9	
10	4	
11	4	
Total	100	

1. [19 points] The graphs of the functions $f(x)$ and $g(x)$ are shown below.





Note that the graph of $f(x)$ is linear for $x < -2$ and $x > 2$, and $g(x)$ is linear on $-3 < x < 1$ and $1 < x < 3$.

For each of the following parts, find the given limit. If any of the quantities do not exist (including the case of limits that diverge to ∞ or $-\infty$), write DNE. If the limit cannot be found based on the information given, write NOT ENOUGH INFO. *You do not need to show any work.*

a. [2 points] Find $\lim_{x\to-1} f(x)$.

Answer: 2

b. [2 points] Find $\lim_{t\to2^-} 2(f(t)-3)$.

Answer: 4

c. [2 points] Find $\lim_{x\to1} f(x)g(x)$.

Answer: **DNE**

d. [2 points] Find $\lim_{x\to\infty} f(e^{-x})$.

Answer: 1

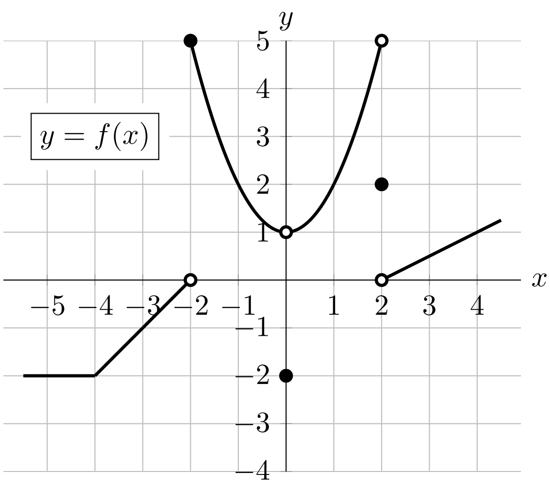
e. [2 points] Find $\lim_{x\to2^+} g^{-1}(x)$.

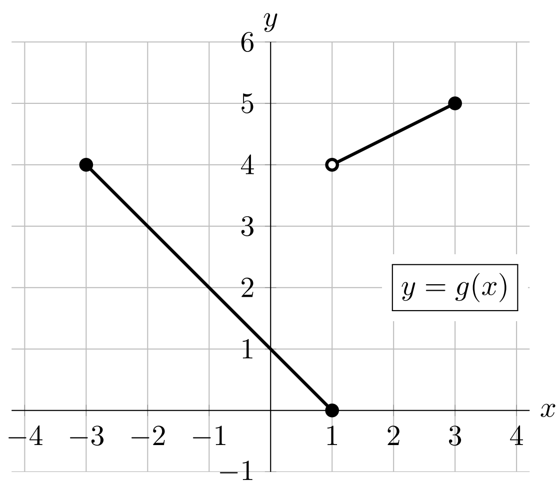
Answer: -1

f. [2 points] Find $\lim_{h\to0} \frac{f(3+h)-f(3)}{h}$.

Answer: 0.5

The graphs of the functions $f(x)$ and $g(x)$ are included here for your convenience.





g. [3 points] Find all the values of x with $-5 < x < 4$ at which the function $f(x)$ is <u>not</u> continuous.

Answer: $-2, 0, 2$

h. [2 points] What is the range of $y = g(x)$?

Answer: **[0,5]**

i. [2 points] For which of the following values of x is $f'(x) > 0$? Circle all that apply.

$x = -5$ $x = -1$ $\boxed{x = 1.5}$ $\boxed{x = e}$ NONE OF THESE

2. [5 points] Let

$$K(p) = (1 + \cos(p))^{1+2p}.$$

Use the limit definition of the derivative to write an explicit expression for $K'(4)$. *Your answer should not involve the letter K. Do not attempt to evaluate or simplify the limit. Please write your final answer in the answer box provided below.*

Answer: $K'(4) = \boxed{\lim_{h\to 0} \frac{(1+\cos(4+h))^{1+2(4+h)} - (1+\cos(4))^{1+2(4)}}{h}}$

3. [10 points] A group of students planted a pine tree and an oak tree alongside the Diag. Let $P(t)$ and $O(t)$ be the height (in feet) of the pine and the oak t years after they were planted, where

$$P(t) = 170 - 165A^{-0.02t} \qquad \text{and} \qquad O(t) = \frac{140}{2 + 100e^{-0.3t}}$$

where $A > 1$ is a constant. *For this problem, your answers should be in exact form or accurate up to the first two decimal places.*

a. [2 points] How tall (in feet) were each of the trees when they were planted?

Answer: Pine: 5 Oak: $\frac{140}{102} \approx 1.372$

b. [4 points] Ten years after the trees were planted, the height of the pine was 38 ft. Find the value of A. *Find your answer algebraically and show all your work.*

Answer:

$$\begin{aligned} 170 - 165A^{-0.2} &= 38 \\ A^{-0.2} &= \frac{132}{165} = 0.8 \\ A &= (0.8)^{-5} \end{aligned}$$

Answer: $A =$ $(0.8)^{-5} \approx 3.051$

c. [4 points] How many years after being planted does it take the oak to be 38 ft? *Find your answer algebraically and show all your work.*

Answer:

$$\begin{aligned} \frac{140}{2 + 100e^{-0.3t}} &= 38 \\ 140 &= 76 + 3800e^{-0.3t} \\ e^{-0.3t} &= \frac{64}{3800} \\ -0.3t &= \ln\left(\frac{64}{3800}\right) \\ t &= -\frac{1}{0.3}\ln\left(\frac{64}{3800}\right). \end{aligned}$$

Answer: $-\frac{1}{0.3}\ln\left(\frac{64}{3800}\right) \approx 13.612$ years

4. [10 points] After the students planted the pine and the oak, the university has been monitoring the growth and health of the trees. Fifteen years after being planted, an invasion of cankerworms (a type of caterpillar) is found on the oak. It is predicted that the number of cankerworms (in hundreds) in the oak s weeks after the pest was detected is given by

$$C(s) = 2e^{0.35s}.$$

a. [2 points] By what percent is the population of cankerworms expected to grow every week?

Answer:
Since $b = e^{0.35}$, then $r = e^{0.35} - 1$. The population grows by $100r\% = 100(e^{0.35} - 1)\%$ every week.

Answer: The population grows by $\underline{100(e^{0.35} - 1)\% \approx 41.9\%}$ every week

b. [3 points] Let $F(m)$ be the number of cankerworms in the oak (in <u>thousands</u>) m <u>days</u> after the pest was detected. Find a formula for $F(m)$ in terms of m only.

Answer: $F(m) = \underline{0.2e^{0.05m}}$

c. [5 points] A population of weevils (another insect) invades the pine. It is estimated that the population of weevils increases by 44 percent every 2 weeks. How many weeks does it take for the population of weevils to triple? *Show all your work and round your answer to the nearest week.*

Answer:
If the population of weevils is given by $W(t) = ab^t$, then the fact that it increases by 44 percent every 2 weeks yields $W(2) = 1.44W(0)$. In other words

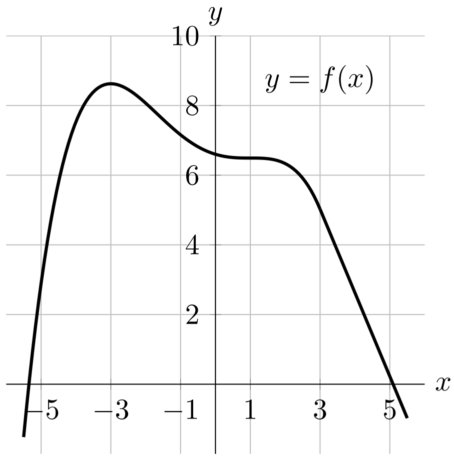
$$ab^2 = 1.44a$$
$$b = \sqrt{1.44} = 1.2.$$

The time T it takes the population to triple satisfies $W(T) = 3W(0)$. Hence

$$a(1.2)^T = 3a \qquad (1.2)^T = 3.$$
$$T\ln(1.2) = \ln(3) \qquad T = \frac{\ln(3)}{\ln(1.2)}$$

Answer: $\frac{\ln(3)}{\ln(1.2)} \approx 6$ **weeks**

5. [7 points] A portion of the graph of the function $f(x)$ is shown below. Note that $f(x)$ is linear for $x > 3$.



a. [4 points] Let the quantities I–V be defined as follows:

I. The number 0.

II. $\dfrac{f(-5) - f(2)}{-5 - 2}$.

III. $f'(-5)$.

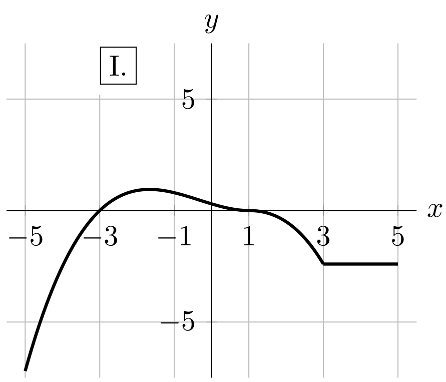
IV. The slope of the secant line between the points on the graph at $x = -3$ and $x = 5$.

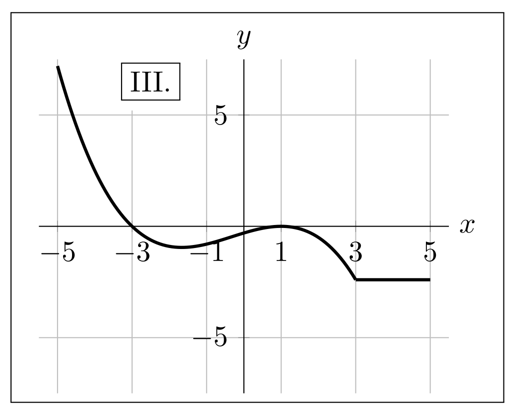
V. The slope of the tangent line at $x = 4$.

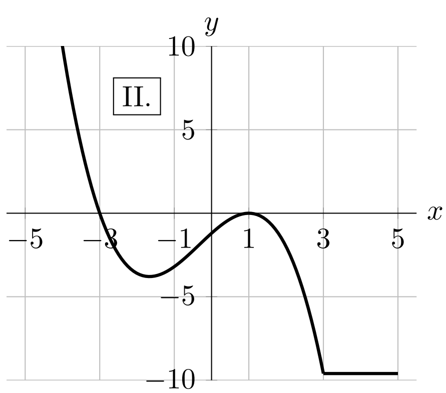
Rank the quantities in order from least to greatest by filling in the blanks below with the options I–V. *You do not need to show your work.*

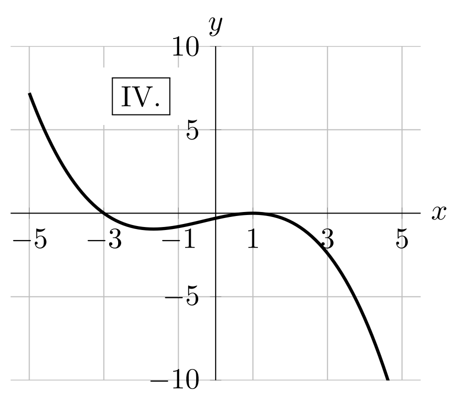
V < **IV** < **0** < **II** < **III**

b. [3 points] There are four graphs below. Circle the <u>one</u> graph that could be the graph of the <u>derivative</u> of $f(x)$. *Note that the graphs are not all drawn at the same scale.*









6. [11 points] A company designs chambers whose interior temperature can be controlled. Their chambers come in two models: Model A and Model B.

a. [5 points] The temperature in Model A goes from its minimum temperature of -3°C to its maximum temperature of 15°C and returning to its minimum temperature three times each day. The temperature of this chamber at 10 am is 15°C. Let $A(t)$ be the temperature (in °C) inside this chamber t hours after midnight. Find a formula for $A(t)$ assuming it is a sinusoidal function.

The amplitude of $A(t)$ is $\frac{15-(-3)}{2} = 9$. The period is 8 hours and the midline is $y = \frac{15+(-3)}{2} = 6$. We know that $A(10) = 15$ a maximum value of $A(t)$. Then
$A(t) = 9\cos\left(\frac{2\pi}{8}(t-10)\right) + 6$

Answer: $A(t) =$ $9\cos\left(\frac{2\pi}{8}(t-10)\right) + 6$

b. [6 points] Let $B(t)$ be the temperature (in $°C$) inside Model B t hours after midnight, where

$$B(t) = 5 - 3\cos\left(\frac{3}{7}t + 1\right).$$

Find the two smallest positive values of t at which the temperature in the chamber is 6°C. Your answer must be found algebraically. *Show all your work and give your answers in exact form.*

$$5 - 3\cos\left(\frac{3}{7}t + 1\right) = 6 \qquad \cos\left(\frac{3}{7}t + 1\right) = -\frac{1}{3}.$$

$$\frac{3}{7}t + 1 = \cos^{-1}\left(-\frac{1}{3}\right) \qquad \frac{3}{7}t + 1 = 2\pi - \cos^{-1}\left(-\frac{1}{3}\right)$$

$$t = \frac{7}{3}\left(\cos^{-1}\left(-\frac{1}{3}\right) - 1\right) \qquad t = \frac{7}{3}\left(2\pi - \cos^{-1}\left(-\frac{1}{3}\right) - 1\right)$$

7. [10 points] Two housecats, Jasper and Zander, escape from their house at the same time and travel along a straight line between their house and a tree. Let $J(t)$ (respectively $Z(t)$) be Jasper's (respectively Zander's) distance, in feet, from the tree t seconds after escaping. The table below shows some of the values of $J(t)$ and $Z(t)$. Assume that $J(t)$ is invertible.

t	6	17	22	31	37
$J(t)$	41	33	21	14	2
$Z(t)$	39	32	31	36	43

a. [2 points] What is Jasper's average velocity for $6 \le t \le 22$? *Be sure to include units.*

Answer: $\frac{21-41}{22-6} = -\frac{20}{16} = -\frac{5}{4} = -1.25$ **ft/sec.**

b. [2 points] Estimate $Z'(31)$. *Remember to show your work.*

You can approximate it using either:

- Average rate of change in $[31, 37]$: $\dfrac{43-36}{37-31} = 7/6$.
- Average rate of change in $[22, 31]$: $\dfrac{36-31}{31-22} = 5/9$.
- Averaging these two: $\frac{1}{2}(7/6 + 5/9) = 0.86\overline{1}$, or
- Average rate of change in $[22, 37]$: $\dfrac{43-31}{37-33} = 0.8$.

Answer: (picking one) 7/6

c. [3 points] Circle the one statement below that is best supported by the equation

$$Z(J^{-1}(8) - 4) = 34.$$

i. 34 seconds after escaping, Zander is 4 feet closer to the tree than Jasper was 8 seconds after escaping.

ii. $\boxed{\text{Four seconds before Jasper is 8 feet from the tree, Zander is 34 feet from the tree.}}$

iii. When Jasper is 4 feet further from the tree than he was 8 seconds after escaping, Zander is 34 feet from the tree.

iv. When Jasper is 4 feet closer to the tree than he was 8 seconds after escaping, Zander is 34 feet from the tree.

v. Four seconds after Jasper is 8 feet from the tree, Zander is 34 feet from the tree.

d. [3 points] Circle the one statement below that is best supported by the equation

$$(J^{-1})'(3) = -0.2.$$

i. In the third second after leaving the house, Jasper travels about 0.2 feet.

ii. When Jasper is 3 feet from the tree, he is traveling about 0.2 feet/second slower than he was one foot earlier.

iii. Jasper gets about 1.5 feet closer to the tree during the third second after leaving the house.

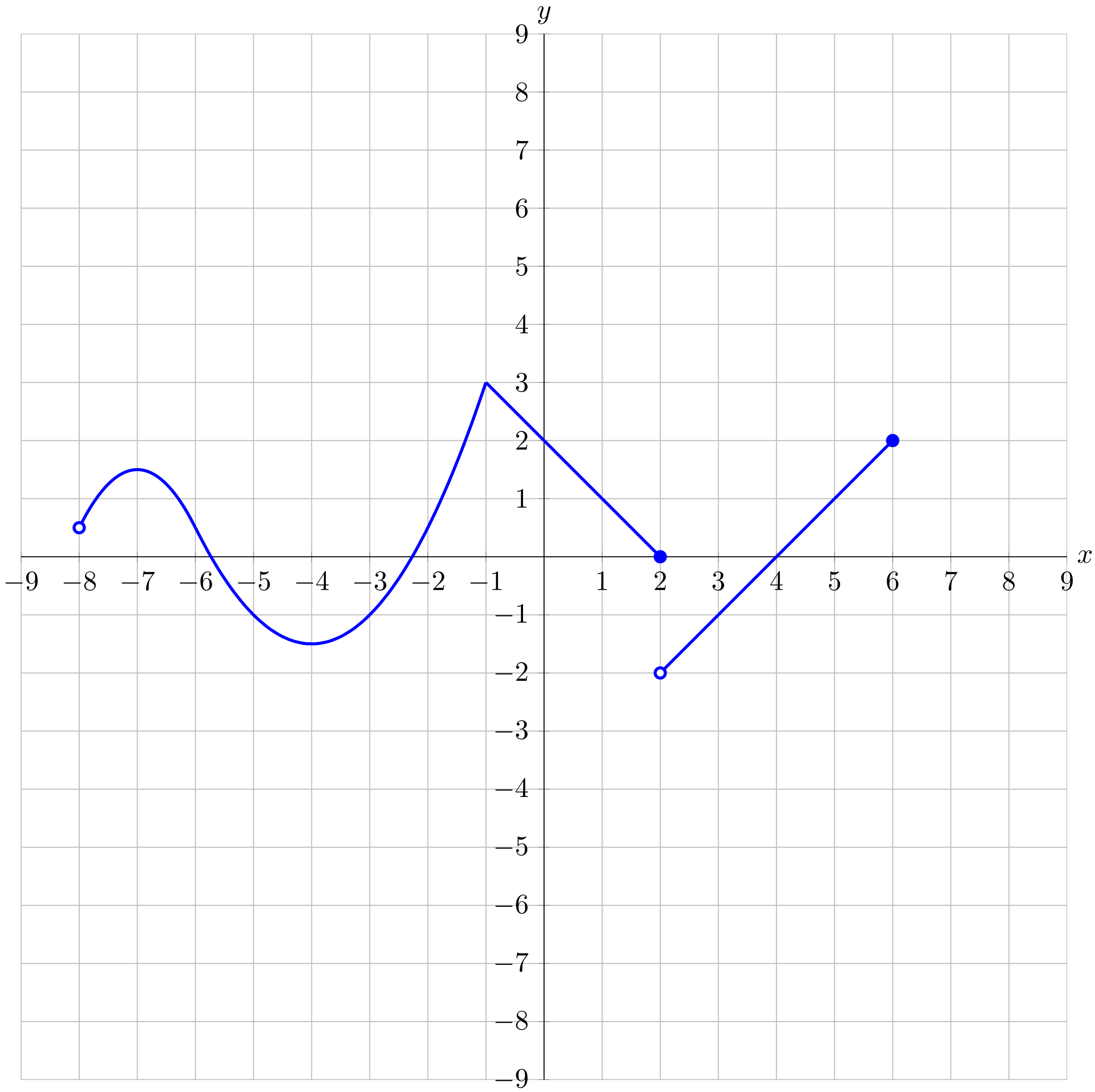
iv. $\boxed{\text{It takes Jasper about one-tenth of a second to go from 3 feet to 2.5 feet from the tree.}}$

v. One-half of a second before Jasper was 3 feet from the tree, he was about 2.9 feet from the tree.

8. [11 points] On the axes provided below, sketch the graph of a single function $y = f(x)$ satisfying all of the following conditions:

- The domain of $f(x)$ is the interval $-8 < x \leq 6$.
- $f(x)$ is continuous for all x in the interval $-8 < x < -2$.
- $f'(-7) = 0$.
- $f(x)$ is decreasing and concave up for all x in the interval $-6 < x < -4$.
- The average rate of change of $f(x)$ is equal to 0.5 between $x = -5$ and $x = -2$.
- $f(0) = 2$ and $f'(0) = -1$.
- $\lim_{x \to 2^-} f(x) = f(2)$ and $\lim_{x \to 2^+} f(x) < \lim_{x \to 2^-} f(x)$.
- $f(x)$ has constant rate of change on the interval $3 \leq x \leq 6$.

Make sure that your graph is large and unambiguous.



9. [9 points] A pharmaceutical company just released a new medication to reduce the cold symptoms in children between 18 months old and 12 years of age. Let $D(z)$ be the dose (in ounces) recommended for a child that is z years old. A table with some values of $D(z)$ is shown below.

z	1.5	3	5	8	10	12
$D(z)$	2	5.2	8.6	11.4	14.5	20.2

a. [3 points] Find a formula for $D(z)$ on $3 \leq z \leq 5$ assuming it is a linear function in this interval.

Using the points $(3, 5.2)$ and $(5, 8.6)$, we can find the slope $m = \dfrac{8.6 - 5.2}{5 - 3} = 1.7$. Applying the point slope formula for the linear function we get $D(z) = 5.2 + 1.7(z - 3)$.

Answer: $D(z) =$ $5.2 + 1.7(z - 3)$ **or** $1.7z + 0.1$

b. [3 points] Suppose that $D(z)$ is invertible. Give a practical interpretation of the equation

$$D^{-1}(9) = 6.5.$$

Answer: A child whose recommended dose is 9 ounces is six and a half years old.

c. [3 points] Below is the first part of a sentence that will give a practical interpretation of the equation $D'(2) = 1.2$ in the context of this problem. Complete the sentence so that the practical interpretation can be understood by someone who knows no calculus. Be sure to include the appropriate units in your answer.

As the age of an child increases from 2 years to 25 months, the recommended dose increases by about 0.1 ounces.

10. [4 points] Find <u>all</u> real numbers B and positive integers k such that the rational function

$$H(x) = \frac{9 + x^k}{16 - Bx^3}$$

satisfies the following two conditions:

- $H(x)$ has a vertical asymptote at $x = 2$
- $\lim_{x\to\infty} H(x)$ exists.

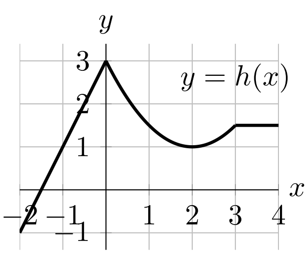
If no such values exist, write NONE.

Justification: In order for $\lim_{x\to\infty} H(x)$ to exist, the degree of the polynomial in the numerator has to be smaller or equal to 3 (the degree of the polynomial in the denominator). In order for $x = 2$ to be a vertical asymptote, you need the denominator to be zero. Hence $16 - B(2^3) = 16 - 8B = 0$ which requires $B = 2$. In this case $H(x) = \frac{9 + x^k}{16 - 2x^3}$ with $k = 1, 2$ or 3. Since $9 + 2^k \neq 0$, then $H(x)$ has a vertical asymptote at $x = 2$ when $B = 2$.

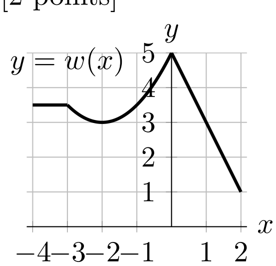
Answer: $B =$ **2** **Answer:** $k =$ **1, 2, or 3**

11. [4 points] A part of the graph of a function $h(x)$ is given below.

In each of the following parts, the corresponding portion of the graph of a function obtained from h by one or more transformations is shown, together with a list of possible formulas for that function. In each case choose the one correct formula for the function shown. *Note that the graphs are not all drawn at the same scale.*



a. [2 points]



A. $h(-x) - 2$

B. $-h(x) - 2$

C. $-h(x) + 2$

D. $\boxed{h(-x) + 2}$

E. $-h(x + 2)$

F. $-h(x - 2)$

G. $-h(-x + 2)$

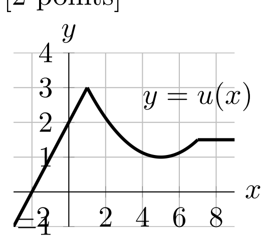
H. $-h(-x - 2)$

I. $h(-x + 2)$

J. $h(-x - 2)$

K. NONE OF THESE

b. [2 points]



A. $h(0.5x + 1)$

B. $h(0.5x - 1)$

C. $\boxed{h(0.5(x - 1))}$

D. $h(0.5(x + 1))$

E. $h(0.5(x - 0.5))$

F. $h(2x - 1)$

G. $h(2x + 1)$

H. $h(2(x - 1))$

I. $h(2(x + 1))$

J. $h(0.5(x + 0.5))$

K. NONE OF THESE