Math 115 — Second Midterm — March 22, 2017

EXAM SOLUTIONS

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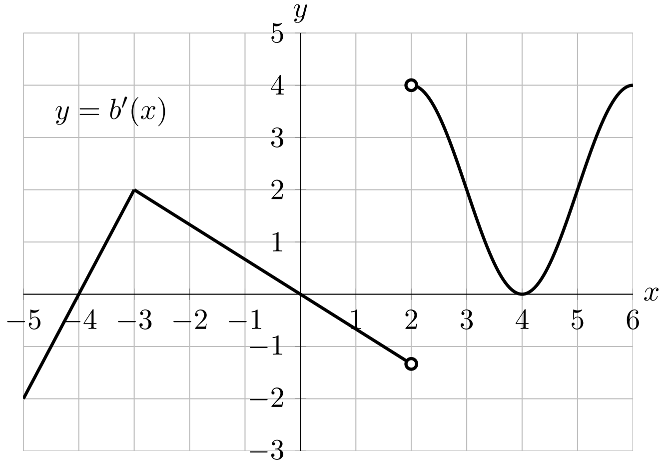
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 15 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	10	
3	9	
4	10	
5	12	

Problem	Points	Score
6	9	
7	9	
8	8	
9	13	
10	10	
Total	100	

1. [10 points] The graph of a portion of the **derivative** of $b(x)$ is shown below. Assume that $b(x)$ is defined and continuous on $[-5, 6]$.



In the following questions, circle **all** correct solutions.

a. [2 points] At which of the following values of x does $b(x)$ appear to have a critical point?.

$\boxed{x = -4}$ $x = -3$ $\boxed{x = 2}$ $x = 3$ NONE OF THESE

b. [2 points] At which of the following values of x does $b(x)$ attain a local minimum?

$\boxed{x = -4}$ $x = 0$ $\boxed{x = 2}$ $x = 4$ NONE OF THESE

c. [2 points] At which of the following values of x does $b(x)$ appear to have an inflection point?

$\boxed{x = -3}$ $x = 2$ $x = 3$ $x = 5$ NONE OF THESE

d. [2 points] On which interval(s) are the hypotheses of the Mean Value Theorem true for $b(x)$?

$\boxed{[-4,-2]}$ $\boxed{[-2,2]}$ [1, 4] [-5,6] NONE OF THESE

e. [2 points] For what values of x is $b(x)$ concave up? Write your answer using inequalities or interval notation.

Answer: $(-5, -3) \cup (4, 6)$

2. [10 points] Let $R(x)$ be a polynomial whose first and second derivatives are given below.

$$R'(x) = (x-1)^7(x+2)^4 \qquad \text{and} \qquad R''(x) = (11x+10)(x-1)^6(x+2)^3$$

a. [6 points] Find the x-coordinates of the inflection points of $R(x)$. Use calculus to find and justify your answers, and show enough evidence to demonstrate that you have found them all. Write NONE if the function $R(x)$ has no points of inflection.

Solution: Potential inflection points: Since $R''(x)$ is defined for all x, we want all the solutions to $R''(x) = 0$, that is $x = -\frac{10}{11}$, $x = 1$ and $x = -2$.
Looking at the signs of $R''(x)$ around these points:

$$\begin{aligned} R''(-3) &= (-)(+)(-) = + \\ R''(-1) &= (-)(+)(+) = - \\ R''(0) &= (+)(+)(+) = + \\ R''(2) &= (+)(+)(+) = + \end{aligned}$$

OR you can compute the values of R"(x) around these points

$$\begin{aligned} R''(-3) &= 94208 \\ R''(-1) &= -64 \\ R''(0) &= 80 \\ R''(2) &= 2048 \end{aligned}$$

Since the sign of $R''(x)$ only changes at $x = -2$ and $x = -\frac{10}{11}$ then the inflection points of $R(x)$ are at $x = -2, -\frac{10}{11}$

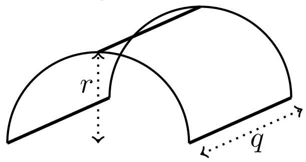
b. [4 points] Find the quadratic approximation $G(x)$ of $R(x)$ at the point $(-1, 5)$ on the graph of $R(x)$. Show all your work.

Solution: $G(x) = R(-1) + R'(-1)(x+1) + \dfrac{R''(-1)}{2}(x+1)^2$ where

$$\begin{aligned} R(-1) &= 5 \qquad \text{since } (-1,5) \text{ is a point on the graph of } R(x). \\ R'(-1) &= (-1-1)^7(-1+2)^4 = -128 \\ R''(-1) &= (11(-1)+10)(-1-1)^6(-1+2)^3 = -64 \end{aligned}$$

Hence $G(x) = 5 - 128(x+1) - 32(x+1)^2$

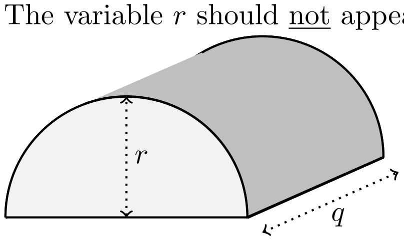
3. [9 points] Duncan's person is making him a new tent in the shape of half a cylinder. She plans to use wire to make the tent frame. This will consist of two semicircles of radius r (measured in inches) attached to three pieces of wire of length q (also measured in inches), as shown in the diagram below. She has 72 inches of wire to use for this.



a. [4 points] Find a formula for r in terms of q.

Solution: The amount of wire S used on the one semicircle of radius r is given by $S = \frac{1}{2}(2\pi r)$ inches. For the rest of the tent, she uses $3q$ inches of wire. Since she used 72 inches of wire to build the tent, we have $r = \dfrac{72 - 3q}{2\pi}$.

b. [2 points] Let $V(q)$ be the volume (in cubic inches) of the space inside the tent after the fabric is added, given that the total length of wire is 72 inches and the length of the tent is q inches. (Recall that the tent shape is half of a cylinder.) Find a formula for $V(q)$. The variable r should not appear in your answer.



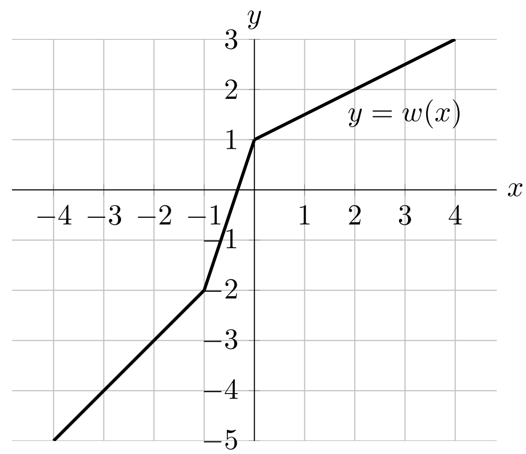
Solution: The volume of enclosed by the tent V is the volume of a half cylinder. In this case $V = \frac{1}{2}\pi r^2 h$, where r is the radius of the semicircular lateral face and h is the length of the tent. In our case $h = q$ and using our answer from part **a** we obtain

$$V = \frac{1}{2}\pi r^2 h = \frac{\pi q}{2}\left(\frac{72 - 3q}{2\pi}\right)^2$$

c. [3 points] In the context of this problem, what is the domain of $V(q)$?

Solution: The length q of the tent cannot be negative and it has to be smaller than the total amount of wire used 72 inches, then $0 < q < 72$. On the other hand, the more wire you use building the length of the tent q the smallest the radius r will be. If we set our expression for r in terms of q from **a.** to 0: $\dfrac{72 - 3q}{2\pi} = 0$, we obtain $3q = 72$ and therefore $q = 24$. Hence the domain of $V(q)$ is all values of q that satisfy $0 < q < 24$.

4. [10 points] A portion of the graph of the function $w(x)$ is shown below.



For each of the parts below, find the value of the given quantity. If there is not enough information provided to find the value, write NOT ENOUGH INFO. If the value does not exist, write DOES NOT EXIST. You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown. All your answers must be in **exact** form.

a. [2 points] Let $k(x) = w^{-1}(x)$. Find $k'(-1.5)$.

Solution: $k'(x) = \dfrac{1}{w'(w^{-1}(x))}$ so $k'(-1.5) = \dfrac{1}{w'(w^{-1}(-1.5))} = \dfrac{1}{w'(-5/6)} = \dfrac{1}{3}$

Answer: $k'(-1.5) = \dfrac{1}{3}$

b. [2 points] Let $h(u) = \ln(3w(u))$. Find $h'(1)$.

Solution: $h'(u) = \dfrac{1}{3w(u)} 3w'(u)$ so $h'(1) = \dfrac{1}{3w(1)} \cdot 3w'(1) = \dfrac{w'(1)}{w(1)} = \dfrac{1/2}{3/2} = \dfrac{1}{3}$.

Answer: $h'(1) = \dfrac{1}{3}$

c. [2 points] Let $n(x) = \dfrac{w(x)}{1 - x^2}$. Find $n'(-2)$.

Solution: $n'(x) = \dfrac{w'(x)(1-x^2) - w(x)(-2x)}{(1-x^2)^2}$ so

$$n'(-2) = \frac{w'(-2)(1-(-2)^2) - w(-2)(-2 \cdot -2)}{(1-(-2)^2)^2} = \frac{(1)(-3) - (-3)(4)}{3^2} = \frac{9}{9}$$

Answer: $n'(-2) = 1$

d. [2 points] Let $s(x)$ be the exponential function $s(x) = 4^{w(x)}$. Find $s'(2)$.

Solution: $s'(x) = \ln(4) \cdot 4^{w(x)} w'(x)$ so $s'(2) = \ln(4) \cdot 4^{w(2)} \cdot w'(2) = \ln(4) \cdot 4^2 \cdot \frac{1}{2}$

Answer: $s'(2) = 8\ln(4)$

e. [2 points] Let $p(x) = x \cdot w^{-1}(x)$. Find $p'(-1)$.

Solution: $p'(x) = 1 \cdot w^{-1}(x) + x \cdot \dfrac{1}{w'(w^{-1}(x))}$ so $p'(-1) = -\dfrac{2}{3} + -1 \cdot \dfrac{1}{3} = -1$

Answer: $p'(-1) = -1$

5. [12 points] Let

$$f(x) = x(x-4)^{4/5}e^{-x} \qquad \text{and} \qquad f'(x) = \frac{(5-x)(5x-4)e^{-x}}{5\sqrt[5]{x-4}}.$$

Note that the domain of $f(x)$ is $(-\infty, \infty)$.

a. [6 points] Find all values of x at which $f(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: First we look for:

- Values of x for which $f'(x) = 0$:

$$\frac{(5-x)(5x-4)e^{-x}}{5\sqrt[5]{x-4}} = 0$$
$$(5-x)(5x-4)e^{-x} = 0$$
$$5-x = 0 \quad \text{or} \quad 5x-4=0$$
$$x = 5 \quad \text{or} \quad x = 0.8.$$

Notice that $e^{-x} \neq 0$ for all values o x. Hence the only solutions are $x = 0.8$ and $x = 5$.

- Values of x for which $f'(x)$ is undefined: The function $f'(x) = \frac{(5-x)(5x-4)e^{-x}}{5\sqrt[5]{x-4}}$ is undefined when.

$$5\sqrt[5]{x-4} = 0$$
$$\sqrt[5]{x-4} = 0$$
$$x - 4 = 0 \quad \text{which yields} \quad x = 4.$$

Finally we notice that all of these points are in the domain of $f(x)$, hence the critical points of $f(x)$ are $x = 0.8, 4, 5$.

We use the first derivative test to classify all the critical points of $f(x)$ by either finding the signs of $f'(x)$ around the critical points or computing its values:

$$f'(0) = \frac{(+)(-)(+)}{-} = + \quad \text{or} \quad f'(0) \approx 3.031$$
$$f'(1) = \frac{(+)(+)(+)}{-} = - \quad \text{or} \quad f'(1) \approx -0.236$$
$$f'(4.5) = \frac{(+)(+)(+)}{+} = + \quad \text{or} \quad f'(4.5) \approx 0.023$$
$$f'(6) = \frac{(-)(+)(+)}{+} = - \quad \text{or} \quad f'(6) \approx -0.011$$

Hence $x = 0.8, 5$ are local maximums and $x = 4$ is a local minimum.

Answer:

Local max(es) at $x =$ $0.8, 5$ Local min(s) at $x =$ 4

b. [6 points] Find the values of x for which $f(x)$ attains a global maximum and global minimum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all global extrema. Write NONE if appropriate.

Solution:

x	0.8	4	5
$f(x)$	0.911	0	0.033

and

$$\lim_{x\to\infty} x(x-4)^{4/5}e^{-x} = \lim_{x\to\infty} \frac{x(x^{4/5})}{e^x} = \lim_{x\to\infty} \frac{x^{9/5}}{e^x} = 0$$

$$\lim_{x\to-\infty} x(x-4)^{4/5}e^{-x} = \lim_{x\to-\infty} x^{9/5}e^{-x} = -\infty.$$

The last limit follows from the fact that $\lim_{x\to-\infty} x^{9/5} = -\infty$ and $\lim_{x\to-\infty} e^{-x} = \infty$. Hence the function $f(x)$ has a global maximum at $x = 0.8$ and no global minimum.

6. [9 points] A group of biology students is studying the length L of a newborn corn snake (in cm) as a function of its weight w (in grams). That is, $L = G(w)$. A table of values of $G(w)$ is shown below.

w	5	10	15	20	25
$G(w)$	24.5	31.6	38.7	44.7	50
$G'(w)$	2.23	1.58	1.30	1.12	1.05

Assume that $G'(w)$ is a differentiable and decreasing function for $0 < w < 25$.

a. [2 points] Find a formula for $H(w)$, the tangent line approximation of $G(w)$ near $w = 20$.

Solution: The formula for is $H(w) = G(20) + G'(20)(w - 20)$. From the table we get $H(w) = 44.7 + 1.12(w - 20)$.

b. [1 point] Use the tangent line approximation of $G(w)$ near $w = 20$ to approximate the length of a corn snake that weighs 22 grams.

Solution: $G(22) \approx H(22) = 1.12(22 - 20) + 44.7 = 46.94$ cm.

c. [2 points] Is your answer in part (b) an overestimate or an underestimate? Circle your answer and write a sentence to justify it.

Solution:

Circle one: **[Overestimate]** Underestimate CANNOT BE DETERMINED

Justification:
Since $G'(w)$ is a differentiable and decreasing function for $0 < w < 25$, then $G(w)$ is concave down on $0 < w < 25$. Hence the values of the tangent line approximation $H(w)$ will be larger than the actual values of $G(w)$ for $0 < w < 25$.

d. [4 points] In their study of the growth of corn snakes, they found the results of a recent article that states that the average weight w of a corn snake (in grams) t weeks after being born is given by $w = \frac{1}{5}t^2$. Let $S(t) = G\left(\frac{1}{5}t^2\right)$ be the length of a corn snake t weeks after being born. Find a formula for $P(t)$, the tangent line approximation of $S(t)$ near $t = 5$.

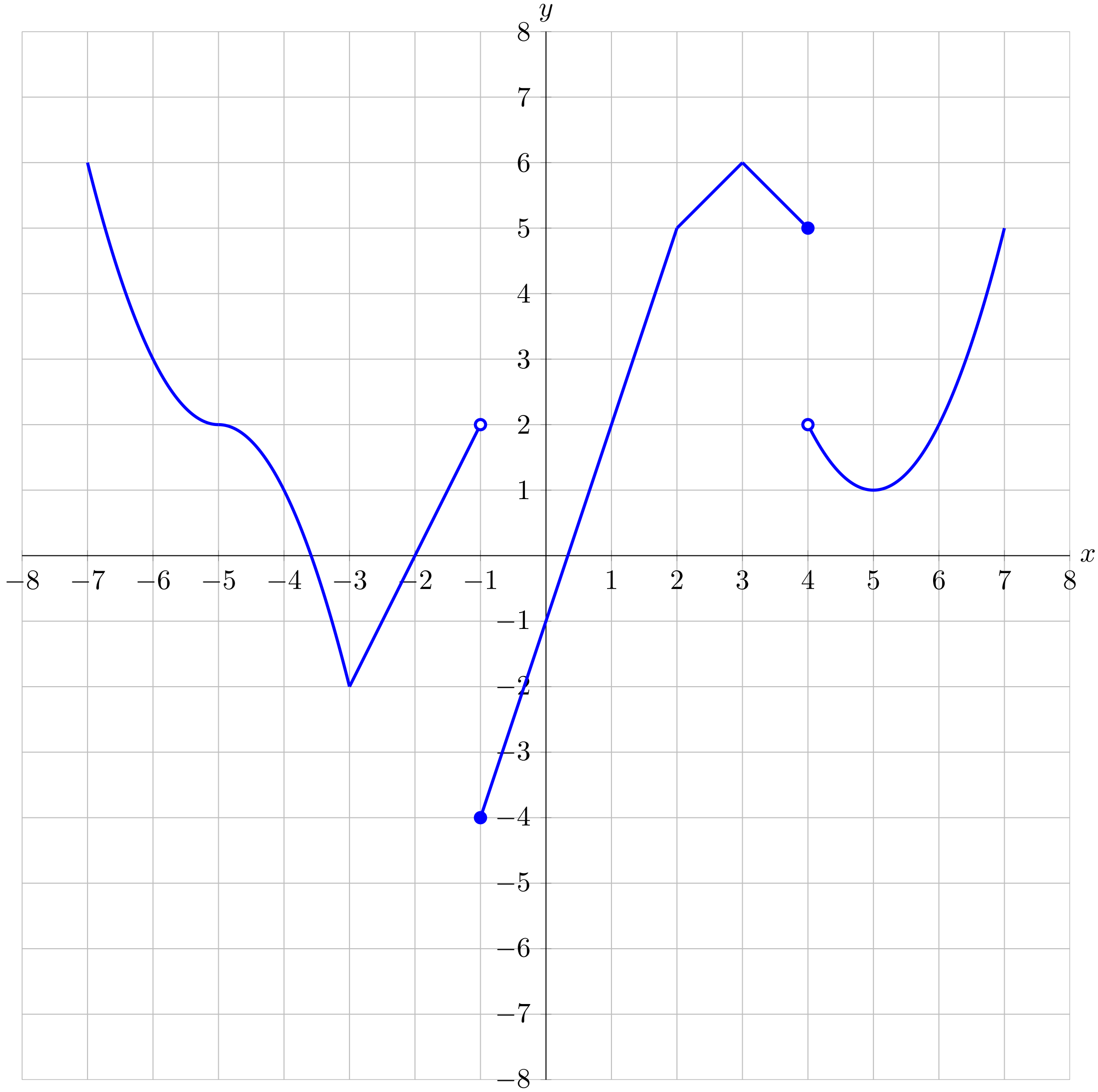
Solution: The formula for the tangent line approximation $P(t)$ is $P(t) = S(5) + S'(5)(t - 5)$. Since $S(t) = G(\frac{1}{5}t^2)$, then $S'(t) = \frac{2}{5}t \cdot G'(\frac{1}{5}t^2)$. Using these formulas we get that $S(5) = G(\frac{1}{5}(5^2)) = G(5) = 25.4$ and $S'(5) = 2 \cdot G'(5) = 4.46$.

Answer: $P(t) =$ $24.5 + 4.46(t - 5) = 4.46t + 2.2$

7. [9 points] On the axes provided below, sketch the graph of a single function $y = h(x)$ satisfying all the following:

- ☐ The function $h(x)$ is defined for $-7 \leq x \leq 7$.
- ☐ $h(x)$ has global maximums at $x = -7$ and $x = 3$.
- ☐ $h(x)$ has an inflection point at $x = -5$.
- ☐ $h(x)$ is continuous at $x = -3$ but not differentiable at $x = -3$.
- ☐ $h(x)$ has a local minimum at $(-1, -4)$ but is not continuous at $x = -1$.
- ☐ $h(x)$ has a critical point at $(2, 5)$ that is neither a local maximum or a local minimum.
- ☐ $h(x)$ satisfies the conclusion of the Mean Value Theorem on [4,7] but not the hypothesis of this theorem.

Make sure that your graph is large and unambiguous.



8. [8 points] At Happy Hives Bee Farm, the population of bees, in thousands, t months after the farm opens, can be modeled by $g(t)$, where

$$g(t) = \begin{cases} 20 + \frac{1}{3}e^{4-t} & \text{for } 0 \le t \le 4 \\ -\dfrac{1}{6}t^3 + \dfrac{9}{4}t^2 - 7t + 23 & \text{for } 4 < t \le 8 \end{cases}$$

and

$$g'(t) = \begin{cases} -\frac{1}{3}e^{4-t} & \text{for } 0 < t < 4 \\ -0.5(t-2)(t-7) & \text{for } 4 < t < 8. \end{cases}$$

a. [6 points] Find the values of t that minimize and maximize $g(t)$ on the interval $[0, 8]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE if appropriate.

Solution: In this case, you need to establish if the function is continuous before using the procedure listed above.

- Continuity of $g(t)$ on $[0, 8]$:
  Since $\lim\limits_{t\to 4^+} g(t) = \lim\limits_{t\to 4^+} -\dfrac{1}{6}t^3 + \dfrac{9}{4}t^2 - 7t + 23 = \dfrac{61}{3}$ and $\lim\limits_{t\to 4^+} g(t) = \dfrac{61}{3} = \lim\limits_{t\to 4^-} g(t) = g(4)$, $g(t)$ is continuous at 4. Both pieces are continuous, so $g(t)$ is continuous on the interval $[0, 4]$.
- Critical points of $g(t)$: Using the formula for $g'(t)$
  - $g'(t) = 0$ on $(0, 4)$: Since $e^{4-t} \ne 0$ for any value of t, then $g'(t) \ne 0$ on $(0, 4)$.
  - $g'(t) = 0$ on $(4, 8)$: In this case $-0.5(t-2)(t-7) = 0$ if $t = 2, 7$. Hence the only solution in $(4, 8)$ is $t = 7$.
  - $g'(t)$ is undefined on $(0, 8)$. Based on the formula for $g'(t)$, the only point where $g'(t)$ could be undefined is $t = 4$. In this case $g'(4)$ does not exist since:
    * $\lim\limits_{h\to 0^-} \dfrac{g(4+h) - g(4)}{h} = -\dfrac{1}{3}e^{4-4} = -\dfrac{1}{3}$
    * $\lim\limits_{h\to 0^+} \dfrac{g(4+h) - g(4)}{h} = -0.5(4-2)(4-7) = 3$

  Hence the critical points of $g(t)$ in $(0, 8)$ are $t = 4, 7$.

Next we make a table to list $g(t)$ at all critical points and endpoints, and choose the values of t corresponding to the min and max from the table.

t	0	4	7	8
$g(t)$	38.199	20.33	27.08	25.67

Answer: Global max(es) at $t =$ **0**

Answer: Global min(s) at $t =$ **4**

b. [2 points] What is the largest population of bees that occurs in the first 8 months the farm is open?

Answer: **38.199 thousand (or** $38,199$**)**

9. [13 points] Let $\mathcal{C}$ be the curve defined by the equation

$$\ln(xy) = x^2.$$

Note that the curve $\mathcal{C}$ satisfies

$$\frac{dy}{dx} = \frac{y(2x^2-1)}{x}.$$

a. [4 points] Exactly one of the following points lies on $\mathcal{C}$. Circle that one point.

$(0,1)$ $(1,0)$ $(1,1)$ $\boxed{(1,e)}$ $(e,1)$ (e,e)

Then find an equation for the line tangent to $\mathcal{C}$ at the point you chose above.

Solution: Plugging $(x,y) = (1,e)$ into $\ln(xy) = \ln(1(e)) = 1$ and $x^2 = 1^2 = 1$.Hence the point $(1,e)$ is in the graph of the equation $\ln(xy) = x^2$.
The slope m of the tangent line at $(1,e)$ is

$$m = \frac{dy}{dx}\Big|_{(1,e)} = \frac{e(2(1)^2-1)}{1} = e.$$

Hence the equation of the tangent line is given by $y = e + e(x-1)$.

b. [4 points] Find all points on $\mathcal{C}$ with a horizontal tangent line. Give your answers as ordered pairs (coordinates). Show your work. Write NONE if no such points exist.

Solution: We know $\mathcal{C}$ has a horizontal tangent line when $\frac{dy}{dx} = 0$. This happens when $y(2x^2-1) = 0$, which happens when $y = 0$ or $2x^2 - 1 = 0$. The latter is true when $x = \pm\frac{1}{\sqrt{2}}$.
Now we plug these values into the equation for $\mathcal{C}$ to find the other coordinates.

1. When $y = 0$, the equation for the x-coordinate is $\ln(x(0)) = 0^2$. This equation has no solutions then there is no point with horizontal tangent lines of the form $(x, 0)$.
2. When $x = \frac{1}{\sqrt{2}}$, then

$$\ln\left(\frac{1}{\sqrt{2}}y\right) = \left(\frac{1}{\sqrt{2}}\right)^2 \quad \text{or} \quad \ln\left(\frac{1}{\sqrt{2}}y\right) = 0.5$$
$$\frac{1}{\sqrt{2}}y = e^{0.5} \quad \text{and} \quad y = \sqrt{2}\sqrt{e} = \sqrt{2e}$$

3. Similarly when $x = -\frac{1}{\sqrt{2}}$

$$\ln\left(-\frac{1}{\sqrt{2}}y\right) = \left(-\frac{1}{\sqrt{2}}\right)^2 \quad \text{yields} \quad y = -\sqrt{2e}$$

Answer: $(x,y) =$ $\left(\frac{1}{\sqrt{2}}, \sqrt{2e}\right), \left(-\frac{1}{\sqrt{2}}, -\sqrt{2e}\right)$

c. [5 points] Consider the curve $\mathcal{D}$ defined by

$$y + 2^x y^4 = 3 - \sin(x^2).$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution:

$$\begin{aligned}
\frac{dy}{dx} + \ln 2 \cdot 2^x y^4 + 2^x \cdot 4y^3 \frac{dy}{dx} &= -2x\cos(x^2) \\
\frac{dy}{dx} + 2^x \cdot 4y^3 \frac{dy}{dx} &= -2x\cos(x^2) - \ln 2 \cdot 2^x y^4 \\
\frac{dy}{dx}\left(1 + 2^x \cdot 4y^3\right) &= -2x\cos(x^2) - \ln 2 \cdot 2^x y^4 \\
\frac{dy}{dx} &= \frac{-2x\cos(x^2) - \ln 2 \cdot 2^x y^4}{1 + 2^x \cdot 4y^3}
\end{aligned}$$

10. [10 points] Some information about a function $f(x)$ is given in the table below.

x	-2	-1	0	1	2	3	4
$f'(x)$	-2	0	-2	0	1	0	-1
$f''(x)$	1	0	0	2	0	0	-2

Assume that $f''(x)$ is continuous on $[-2, 4]$ and that the values of $f'(x)$ and $f''(x)$ are strictly positive or strictly negative between consecutive table entries. You do not need to justify your answers to the following questions.

a. [2 points] Circle all of the intervals on which $f''(x)$ must be negative.

$-2 < x < -1$ $\boxed{-1 < x < 0}$ $0 < x < 1$ $1 < x < 2$

$\boxed{2 < x < 3}$ $\boxed{3 < x < 4}$ None of these

b. [2 points] Circle all of the values of x for which $f(x)$ must have a local minimum.

$x = -1$ $x = 0$ $\boxed{x = 1}$ $x = 2$ $x = 3$ None of these

c. [2 points] Circle all of the values of x for which $f(x)$ must have an inflection point.

$\boxed{x = -1}$ $\boxed{x = 0}$ $x = 1$ $\boxed{x = 2}$ $x = 3$ None of these

d. [2 points] At which value(s) of x does $f(x)$ have a global maximum on $[1, 4]$?

$x = 1$ $x = 2$ $\boxed{x = 3}$ $x = 4$ None of these   Cannot be determined

e. [2 points] At which value(s) of x does $f(x)$ have a global minimum on $[1, 4]$?

$x = 1$ $x = 2$ $x = 3$ $x = 4$ None of these   [Cannot be determined]