Math 115 — Final Exam — April 24, 2017

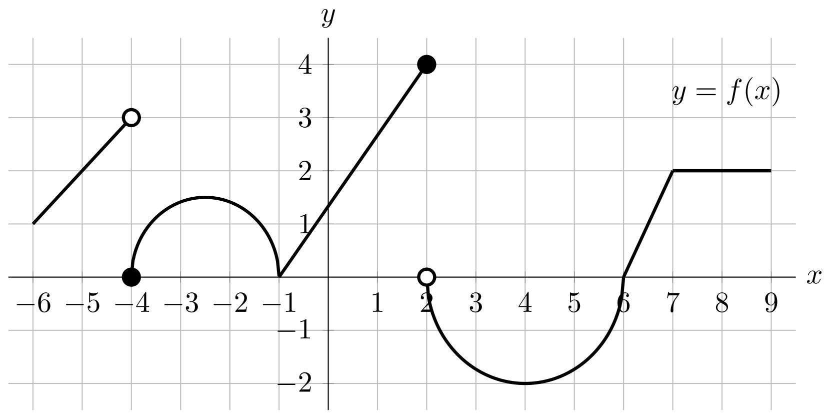
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	9	
3	12	
4	5	
5	9	

Problem	Points	Score
6	11	
7	9	
8	16	
9	9	
10	10	
Total	100	

1. [10 points] The graph of $f(x)$ shown below consists of lines and semicircles.



For the following problems, you do not need to show work. If there is not enough information, write "NEI".

a. [2 points] For which values of $-6 < x < 9$ is the function $f(x)$ discontinuous?

Solution: $x = -4$ and $x = 2$.

b. [2 points] For which values of $0 < x < 9$ does $f(x)$ appear to not be differentiable?

Solution: $x = 2$, 6 and 7.

c. [2 points] Find $\lim\limits_{h\to 0^-} f(-4+h) - f(-4)$.

Solution: $\lim\limits_{h\to 0^-} f(-4+h) - f(-4) = 3$.

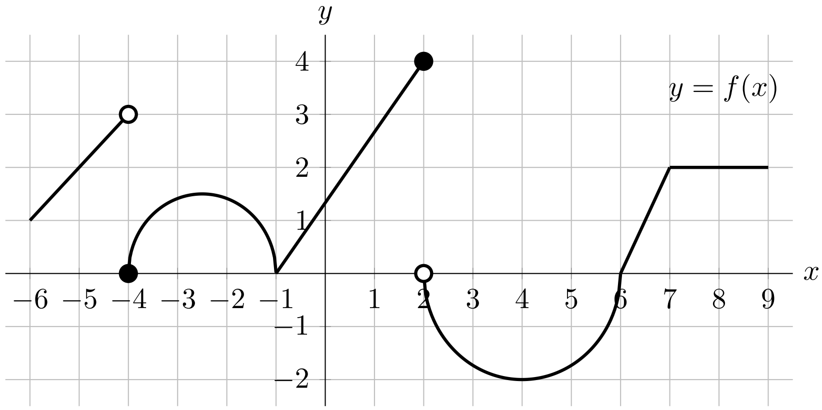
d. [2 points] Find $\lim\limits_{x\to\infty} f\left(\dfrac{2x}{x+1}\right)$.

Solution: $\lim\limits_{x\to\infty} f\left(\dfrac{2x}{x+1}\right) = 4$

e. [2 points] Let $g(x) = \ln(4 + f(x))$. Find $g'(6.5)$.

Solution: $g'(x) = \dfrac{f'(x)}{4+f(x)}$, then $g'(6.5) = \dfrac{f'(6.5)}{4+f(6.5)} = \dfrac{2}{5}$

2. [9 points] The graph of $f(x)$ shown below consists of lines and semicircles.



Use the graph above to calculate the answers to the following questions. Give your answers as exact values. You do not need to show work. If any of the answers can't be found with the information given, write "NEI".

a. [3 points] Find the average value of $f(x)$ on $[-4, 2]$.

Solution: $\frac{1}{6}\int_{-4}^{2} f(x)dx = \frac{1}{6}(\frac{1}{2}\pi(1.5)^2 + \frac{1}{2}(4)(3)) = \frac{1}{6}(\frac{9\pi}{8} + 6) = \frac{9\pi}{48} + 1$

b. [2 points] Find the value of $\int_4^9 |f(z)|dz$.

Solution: $\int_4^9 |f(z)|dz = -\int_4^6 f(z)dz + \int_6^9 f(z)dz = \frac{1}{4}\pi(2)^2 + \frac{1}{2}(3+2)(2) = 5 + \pi$

c. [2 points] Find the value of $4 < T \leq 9$ such that $\int_4^T f(x)dx = 0$.

Solution: We need to find a value of T for which

$$\int_4^T f(x)dx = \int_4^6 f(x)dx + \int_6^T f(x)dx = 0.$$

From the graph $\int_4^6 f(x)dx = -\pi$ and $\int_6^T f(x)dx = \frac{1}{2}((T-6)+(T-7))(2) = 2T - 13$.
Solving for T on $2T - 13 = \pi$, we get $T = \frac{\pi + 13}{2}$.

d. [2 points] Find the value of $\int_{-8}^{-7} f(x+2) + 1\, dx$.

Solution:

$$\int_{-8}^{-7} f(x+2) + 1\, dx = \int_{-6}^{-5} f(x) + 1\, dx = \int_{-6}^{-5} f(x)dx + 1 = 1.5 + 1 = 2.5.$$

3. [12 points] Virgil, Duncan, Jasper and Zander are all watching a toy wind-up mouse move across the floor. Their person places the toy on the floor 2.3 meters away from Virgil, and it moves in a straight line directly away from Virgil at a strictly decreasing velocity. Below are some values of $v(t)$, the velocity of the toy mouse, in meters per second, t seconds after the person places it on the floor, where a positive velocity corresponds to the toy moving away from Virgil.

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$v(t)$	3.19	2.39	1.86	1.43	1.11	0.86	0.54	0.42	0.11

a. [4 points] Estimate the value of $\int_{0.25}^{1.75} v(t)\,dt$ using a left-hand Riemann sum with $\Delta t = 0.5$. Be sure to write down all the terms in your sum. Is your answer an over- or underestimate?

Solution: Left hand sum$=0.5(2.39 + 1.43 + 0.86) = 2.34$.

This is (circle one):

$\boxed{\text{AN OVERESTIMATE}}$ AN UNDERESTIMATE NOT ENOUGH INFORMATION

b. [3 points] How often should the values of $v(t)$ be measured in order to find upper and lower estimates for $\int_{0.25}^{1.75} v(t)\,dt$ that are within 0.1 m of the actual value?

Solution: We can estimate the size of Δt using the formula

$$|v(1.75) - v(0.25)|\Delta t = |0.42 - 2.39|\Delta t = 1.97\Delta t \leq 0.1.$$

This yields $\Delta t \leq \frac{0.1}{0.97} \approx 0.0507$ seconds.

c. [2 points] Find the value of $\int_{0.5}^{1.25} v'(t)\,dt$.

Solution: Using the Fundamental Theorem of Calculus $\int_{0.5}^{1.25} v'(t)\,dt = v(1.25) - v(0.5) = 0.86 - 1.86 = -1$.

d. [3 points] Which of the following represents how much the distance from the toy mouse to Virgil increases during the 2$^{\text{nd}}$ second after it has been placed on the floor? Circle the one best answer.

i. $2.3 - \int_1^2 v(t)\,dt$

ii. $2.3 - \int_1^2 v'(t)\,dt$

iii. $\int_1^2 v(t)dt - \int_0^1 v(t)\,dt$

iv. $\boxed{\int_1^2 v(t)\,dt}$

v. $\int_1^2 v'(t)\,dt$

vi. $v(2) - v(1)$

4. [5 points] Consider the function $Z(w) = \arctan(kw) - (w+1)$ where k is a nonzero constant. Use the limit definition of the derivative to write an explicit expression for $Z'(-2)$. *Your answer should not involve the letter Z. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Solution:

Answer: $Z'(-2) = \boxed{\lim_{h\to 0} \frac{\arctan(k(-2+h)) - ((-2+h)+1) - (\arctan(-2k)+1)}{h}}$

5. [9 points] A cylindrical bar of radius R and length L (both in meters) is put into an oven. As the bar gains temperature, its radius decreases at a constant rate of 0.05 meters per hour and its length increases at a constant rate of 0.12 meters per hour. Fifteen minutes after the bar was put into the oven, its radius and length are 0.4 and 3 meters respectively. At what rate is the volume of the bar changing at that point? *Be sure to include units.*

Solution: The volume of the cylindrical bar is $V = \pi R^2 L$. Differentiating with respect to t, we obtain

$$\frac{dV}{dt} = \pi\left(2R\frac{dR}{dt}L + R^2\frac{dL}{dt}\right).$$

You are given that after 15 minutes: $R = 0.4$, $L = 3$, $\frac{dR}{dt} = -.05$ and $\frac{dL}{dt} = 0.12$. Then

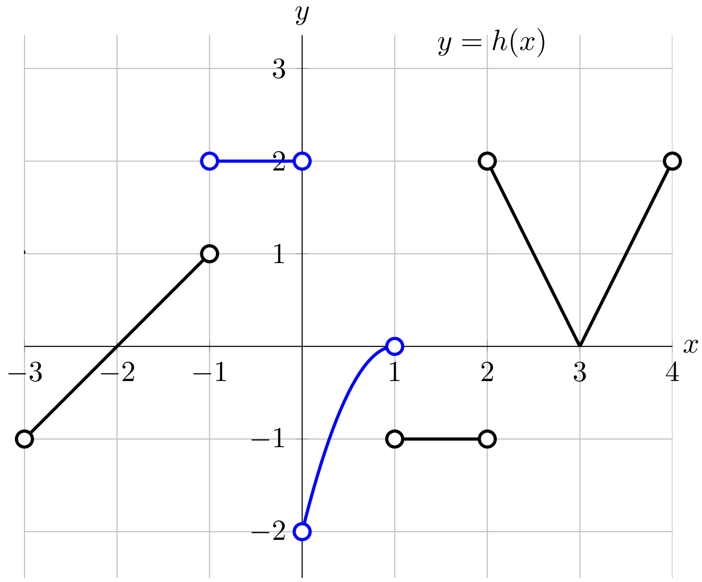
$$\frac{dV}{dt} = \pi\left(2(0.4)(-0.05)(3) + (0.4)^2(0.12)\right) = -0.3166.$$

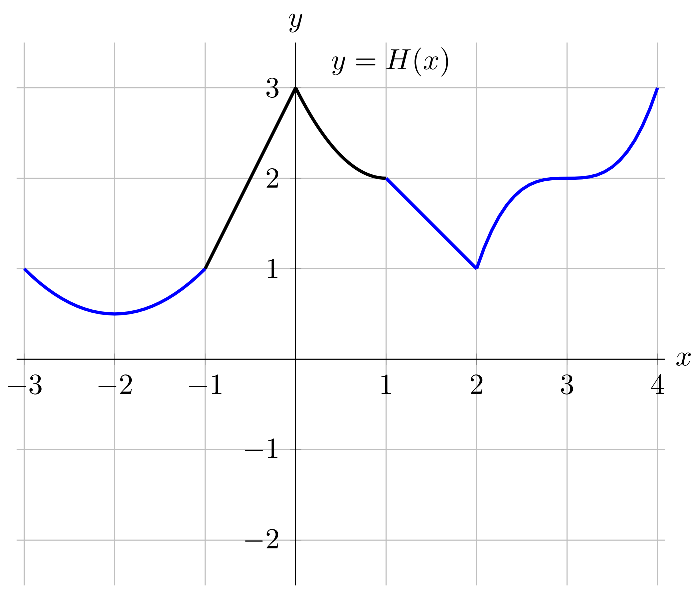
Answer: The volume of the bar is (circle one):

INCREASING $\boxed{\text{DECREASING}}$ NOT ENOUGH INFORMATION

at a rate of 0.3166 m^3 per hour.

6. [11 points] Suppose $h(x)$ is a function and $H(x)$ is an antiderivative of $h(x)$ such that $H(x)$ is defined and continuous on the entire interval $-3 \le x \le 4$. Portions of the graphs of $h(x)$ and $H(x)$ are shown below.





a. [4 points] Use the portions of the graphs shown to fill in the exact values of $H(x)$ in the table below.

x	-3	-2	1	2	4
$H(x)$	1.0	0.5	2.0	1.0	3.0

b. [7 points] Use the axes above to sketch the missing portions of the graphs of <u>both</u> h and H over the interval $-3 \le x \le 4$.
Be sure that you pay close attention to each of the following:

- the values of $H(x)$ you found in part (a) above
- where H is/is not differentiable
- where H and h are increasing, decreasing, or constant
- the concavity of the graph of $y = H(x)$

7. [9 points] Consider the family of functions

$$f(x) = ax^2e^{-bx}$$

where a and b are positive constants. Note that

$$f'(x) = ax(2 - bx)e^{-bx}.$$

a. [4 points] Find the exact values of a and b so that $f(x)$ has a critical point at $(4, e^{-2})$.

Solution: Since $(4, e^{-2})$ is a critical point of $f(x)$, then $f'(4) = 0$ or $0 = 4a(2 - 4b)e^{-4b}$. From this equation we get that $2 - 4b = 0$ (since $a, e^{-4b} > 0$). Then $b = 0.5$. We also know that the point $(4, e^{-2})$ is in the graph of $y = f(x)$, then $e^{-2} = 16ae^{-4b}$. Plugging the value of b, we get $e^{-2} = 16ae^{-2}$. This yields $1 = 16a$, so $a = \dfrac{1}{16}$.

b. [5 points] Using your values of a and b from the previous part, find and classify the local extrema of $f(x)$. Use calculus to find and justify your answers, and be sure to show enough evidence that you have found them all. For each answer blank, write NONE if appropriate.

Solution: With the values found above $f'(x) = \frac{1}{16}x(2 - 0.5x)e^{-0.5x}$. The critical points are found by solving

$$f'(x) = \frac{1}{16}x(2 - 0.5x)e^{-0.5x} = 0$$

In this case, we have $x = 0$ or $2 - 0.5x = 0$ (since $.e^{-0.5x} > 0$). Hence the critical points are $x = 0$ and $x = 4$. To classify them we use the first derivative test:

- $f'(-1) = \dfrac{1}{16}(-1)(2 - 0.5(-1))e^{-0.5(-1)} = -0.257$
- $f'(1) = \dfrac{1}{16}(2 - 0.5)e^{-0.5} = 0.0568$
- $f'(5) = \dfrac{1}{16}(5)(2 - 0.5(5))e^{-0.5(5)} = -0.0128$

OR

- $f'(-1) = (-)(+)(+) = -$
- $f'(1) = (+)(+)(+) = +$
- $f'(5) = (+)(-)(+) = -$

Answer: Local max(es) at $x = 4$ Local min(s) at $x = 0$

8. [16 points] An apple farmer starts harvesting apples on her orchard. They start collecting apples at 6 am. Let $a(t)$ be the total amount of apples (in hundreds of pounds) that have been harvest t hours after 6 am. Some of the values of the invertible function $a(t)$, its derivative $a'(t)$ and an antiderivative function $b(t)$ are shown below.

t	3	4.5	6	7.5	9	10.5	12
$a(t)$	1.5	2	3	4.5	6	6.5	9

t	3	6	9	12
$a'(t)$	0.4	1.2	0.5	1.8

t	3	6	9	12
$b(t)$	6	12.5	25.5	43

a. [2 points] Use the tables to estimate the value of $a''(9)$. Show your work.

Solution: Possible approximations:
$a''(9) \approx \dfrac{1.8-0.5}{12-9} \approx 0.433$, $a''(9) \approx \dfrac{0.5-1.2}{9-6} \approx -.233$ or $a''(9) \approx \dfrac{0.433-.233}{2} = 0.1$

b. [3 points] Find the value of $(a^{-1})'(6)$. What are its units in the context of this problem?

Solution: $(a^{-1})'(6) = \dfrac{1}{a'(a^{-1}(6))} = \dfrac{1}{a'(9)} = \dfrac{1}{0.5} = 2$ hours per hundreds of pounds of apples.

c. [3 points] Use the fact that $a'(10) = 3.2$ to complete the sentence below, including units, to give a practical interpretation in the context of this problem that can be understood by someone who knows no calculus.
The amount of apples harvested between 4 pm and 4:30 pm . . .

Solution: increases by approximately 160 pounds of apples.

d. [3 points] Find the tangent line approximation $S(t)$ of $b(t)$ near $t = 3$.

Solution: $S(t) = b(3) + b'(3)(t-3) = 6 + 1.5(t-3)$.

e. [2 points] Use your answer in **d** to approximate the value of $b(2)$.

Solution: $b(2) \approx S(2) = 6 - 1.5 = 4.5$.

f. [1 point] Is your answer in **e** an overestimate or an underestimate? Circle your answer.

Solution:

OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ NOT ENOUGH INFO

g. [2 points] Let $m(t)$ be an antiderivative of $a(t)$ satisfying $m(9) = -1$. Find $m(3)$.

Solution: We know that two antiderivatives $b(t)$ and $m(t)$ of $a(t)$ satisfy $m(t) = b(t) + C$. Then using $t = 9$ we get that $C = m(9) - b(9) = -1 - 25.5 = -26.5$. Hence $m(3) = b(3) - 26.5 = 6 - 26.5 = -20.5$.

9. [9 points] A Math 115 coordinator is trying to create functions with certain properties in order to test students' understanding of various calculus concepts.

a. [5 points] He wants a function $f(x)$ of the form

$$f(x) = \begin{cases} ax^2 + ax + be^x & \text{for } x < 0 \\ a + 2\cos(x) & \text{for } x \geq 0 \end{cases}$$

where a and b are constants.
Find all value(s) of a and b for which $f(x)$ be differentiable at $x = 0$. Show enough work to justify your answer.

Solution: In order for $f(x)$ to be differentiable at $x = 0$, $f(x)$ has to be continuous. Then

$$\lim_{x\to 0^-} ax^2 + ax + be^x = b = \lim_{x\to 0^+} a + 2\cos(x) = a + 2 = f(0).$$

Then $b = a + 2$. If $f(x)$ is differentiable then

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = 2a(0) + a + be^0 = a + b.$$
$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = -2\sin(0) = 0.$$

Hence $a + b = 0$. Using both equation we obtain $a = -1$ and $b = 1$.

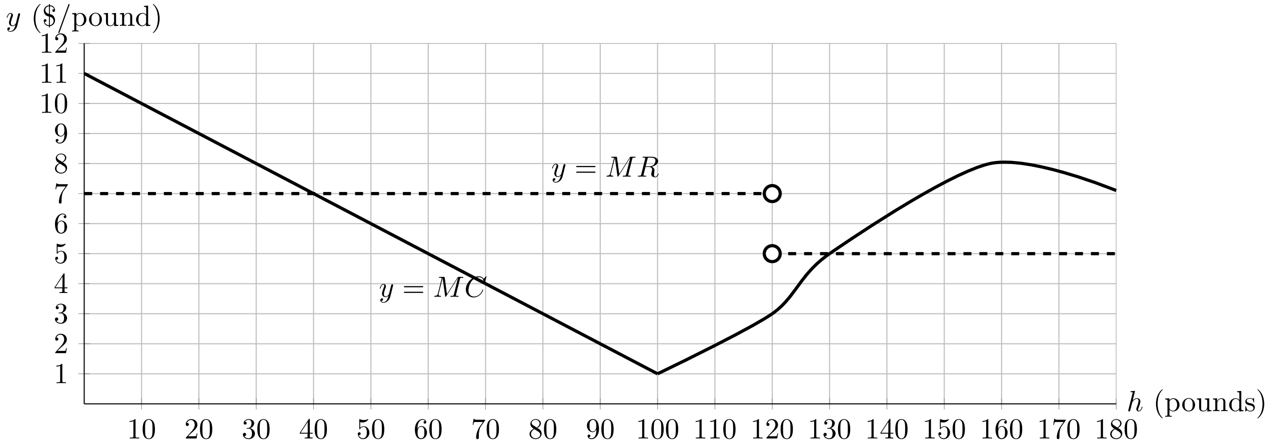
b. [4 points] The coordinator also wants a function $g(x) = cx - e^x$, where c is a constant, so that $g(x)$ has at least one critical point. What condition(s) on c will make this true? Find the x-values of all critical points in this case. Your answer may be in terms of c.

Solution: The function $g(x)$ has critical points at values of x that satisfy

$$g'(x) = c - e^x = 0.$$

Then the only potential critical point is $x = \ln(c)$. This critical point exists only if $c > 0$.

10. [10 points] The Happy Hives Bee Farm sells honey. The graph below shows marginal revenue MR (dashed) and marginal cost MC (solid), in dollars per pound, where h is the number of pounds of honey.



a. [7 points] Use the graph to estimate the answers to the following questions. You do not need to show work. If an answer can't be found with the information given, write "NEI".

i) For what value(s) of h in the interval $[0, 180]$ is the cost function C minimized?

Answer: $h = 0$.

ii) For what value(s) of h in the interval $[0, 180]$ is MC minimized?

Answer: $h = 100$.

iii) For what value(s) of h in the interval $[0, 180]$ is profit maximized?

Answer: $h = 130$.

iv) What are the fixed costs of the farm?

Answer: NEI

v) For what values of h in the interval $[0, 180]$ is the profit function concave up?

Answer: $(0, 100) \bigcup (160, 180)$

b. [3 points] The farm currently sells 20 pounds of honey but is thinking of increasing to 80 pounds of honey.

Solution: The total change in profit from selling 20 to 80 pounds of honey is given by $\int_{20}^{80} MR(q) - MC(q)dq = \frac{1}{2}(2+4)(20) = 60$.

Will this increase or decrease profit? (Circle one.)

$\boxed{\text{INCREASE}}$ DECREASE

By approximately how much will the profit change? 60 dollars.