Math 115 — First Midterm — February 6, 2018

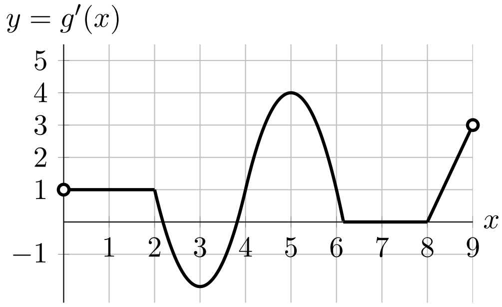
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	14	
3	13	
4	7	
5	5	
6	10	

Problem	Points	Score
7	10	
8	15	
9	12	
10	6	
Total	100	

1. [8 points] The graph of the *derivative* $g'(x)$ of a function $g(x)$ with domain $0 < x < 9$ is shown below.



a. [2 points] On which interval(s) is the function $g(x)$ constant?

Solution: Approximately on the interval $(6.2, 8)$

b. [2 points] On which interval(s) is the function $g(x)$ linear?

Solution: $(0, 2]$ and approximately on the interval $(6.2, 8)$

c. [2 points] On which interval(s) is the function $g(x)$ decreasing?

Solution: Approximately on the interval $(2.25, 3.75)$

d. [2 points] At which value(s) of $0 < x < 9$ is the function $g(x)$ increasing the fastest?

Solution: $x = 5$

2. [14 points] A group of divers recently discovered one of the most submerged caverns in the world. As part of the exploration team, Elena descended into the caverns to take measurements of the temperature and pressure at different depths in the water. Elena started her descent at 8 am and reached the bottom of the caverns at 8:30 am. Let

- $A(t)$ be Elena's depth (in meters) during her descent t minutes after 8 am,
- $B(p)$ be the depth (in meters) at which Elena measures a water pressure of p kPa (kilo-Pascals),
- $C(m)$ be the water temperature (in degrees Celsius) at a depth of m meters.

Assume all these functions are differentiable and invertible.

a. [4 points] Find mathematical expressions that represent each of the sentences below.

(i) The temperature of the water in degrees Celsius when its pressure is 118 kPa.

Solution: $C(B(118))$

(ii) The water pressure, in *Pascals*, 2 meters under the water surface (*1 kPa=1000 Pascals*).

Solution: $1000B^{-1}(2)$

b. [3 points] At 8:02am, Elena started recording all the data that she was measuring. Let $F(x)$ be Elena's depth (in meters) x *seconds* after she *started recording data*. Find a formula for $F(x)$ in terms of any of the functions A, B or C.

Solution: $F(x) = A\left(2 + \dfrac{x}{60}\right)$

c. [4 points] Complete the following sentence using the fact $C'(30) = -0.8$

Solution: As Elena submerges from 29.7 meters to 30 meters below the water surface, the water's temperature ... **decreases by about** 0.24^o **C**.

d. [3 points] Circle the <u>one</u> statement below that is best supported by the equation

$$(B^{-1})'(18) = 1.5.$$

(i) The water pressure increases by 18 kPa for every 1.5 meters that Elena submerges.

(ii) After Elena has submerged 18 meters, she needs to submerge about 1.5 meters more in order for the water pressure to increase by 1 kiloPascal.

(iii) Once Elena has submerged 18 meters, the water pressure increases about 0.75 kPa as she submerges half a meter more.

(iv) Once the water pressure is 18 kPa, to reach a depth where the water pressure is 19 kPa, Elena must submerge about 1.5 meters more.

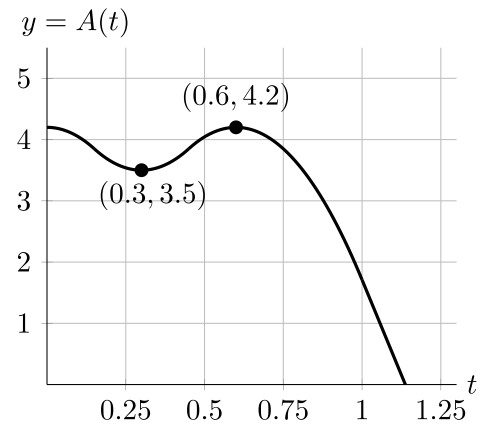
(v) Once Elena has submerged 18 meters, she has to submerge an additional 0.1 meter for the water pressure to increase by approximately 1.5 kPa.

Solution: (iii)

3. [13 points] Tom organizes another meeting of his Science Club, but this time only Anne and John can make it. The meeting is at 2 pm, so they both start walking from their houses to Tom's at 1 pm. At 1:18 pm, Anne realizes she forgot her wallet, so she goes back home to get it before heading over to Tom's house.

Anne's distance in kilometers, $A(t)$, and John's distance in kilometers, $J(t)$, to Tom's house t hours after 1 pm are given by the graph and the table below. Assume that both of them walk along a straight line.

t	0	0.2	0.4	0.5	0.8	0.9
$J(t)$	5.5	4.3	3.2	2.8	0.8	0



a. [1 point] How many kilometers from Tom's house is Anne's house?

Solution: 4.2 km.

b. [2 points] Estimate $J'(0.4)$. Show all your computations. Include units.

Solution: $J'(0.4) \approx \frac{2.8 - 3.2}{0.5 - 0.4} = -\frac{0.4}{0.1} = -4$ kilometers per hour.

c. [3 points] Rank John's average velocity over the time intervals

(I) $0.2 \le t \le 0.4$ (II) $0.5 \le t \le 0.9$ (III) $0.8 \le t \le 0.9$

from least to greatest. Show your work and indicate your final answer by filling in the blanks with I, II, III.

Solution: $[0.2, 0.4]$: $\frac{3.2 - 4.3}{0.2} = -5.5$, $[0.5, 0.9]$: $\frac{0 - 2.8}{0.4} = -7$,

$[0.8, 0.9]$: $\frac{0 - 0.8}{0.1} = -8$ III $\le$ II $\le$ I.

d. [2 points] What was the total distance travelled by Anne?

Solution: distance$=2(0.7) + 4.2 = 5.6$ kilometers.

e. [2 points] At which of the following times was Anne's speed the largest? Circle the correct answer(s).

Solution:

$t = 0.05$ $t = 0.3$ $t = 0.4$ $t = 0.6$ **[$t = 1$]**

f. [3 points] On which of the following intervals is $A(t)$ invertible? Circle the correct answer(s).

Solution:

$[0, 0.6]$ **[$[0.3, 0.6]$]** $[0.1, 0.5]$ **[$[0.6, 1]$]** $[0, 1]$

4. [7 points] The numbers, in thousands, of men $M(t)$ and women $W(t)$ on Logistic Island t years after January 1st, 2015 are given by the formulas

$$M(t) = \frac{150}{5 + 95(1.5)^{-3t}}, \qquad W(t) = \frac{A}{2 + 100e^{-kt}},$$

where $A, k > 0$ are constants.

a. [1 point] What was the population of men on Logistic Island on January 1st, 2015?

Solution: $M(0) = 1.5$ then there were 1500 men in the island at that time.

b. [4 points] When was the number of men on the Logistic Island equal to two thousand? Your answer needs to be *exact*. Show all your work.

Solution:

$$\begin{aligned}
\frac{150}{5 + 95(1.5)^{-3t}} &= 2 \\
150 &= 10 + 190(1.5)^{-3t} \\
(1.5)^{-3t} &= \frac{14}{19} \\
-3t\ln(1.5) &= \ln\left(\frac{14}{19}\right) \\
t &= -\frac{1}{3\ln(1.5)}\ln\left(\frac{14}{19}\right)
\end{aligned}$$

c. [2 points] Knowing that $\lim_{t\to\infty} W(t) = 50$, find the value of A.

Solution: Since $\lim_{t\to\infty} e^{-kt} = 0$ then $\lim_{t\to\infty} W(t) = \frac{A}{2} = 50$. Then $A = 100$.

5. [5 points] On the axes provided below, sketch the graph of a single function $y = f(x)$ satisfying all of the following conditions:

(i) the function $f(x)$ is defined on $-6 < x < 6$ and continuous on $-6 < x < 3$,

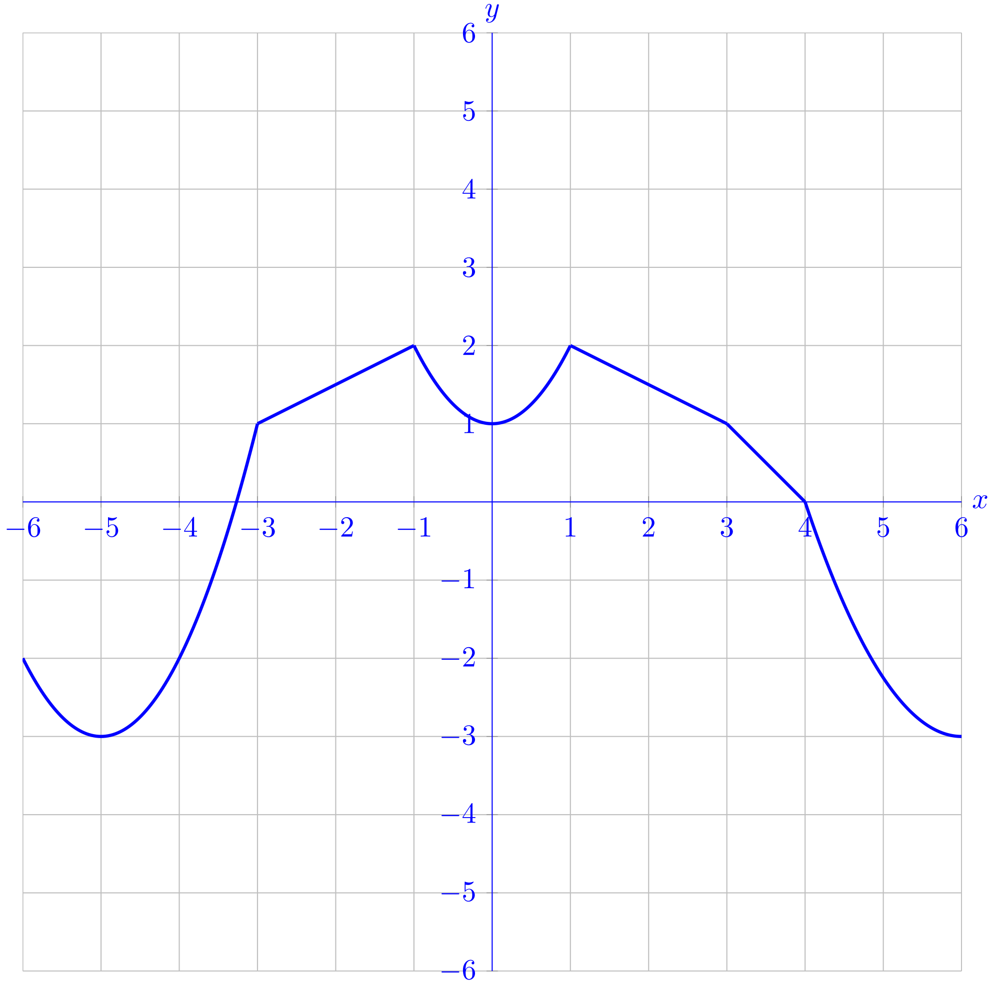
(ii) the average rate of change of $f(x)$ on $[-5, -3]$ is equal to 2,

(iii) $f'(x) = -\frac{1}{2}$ for $1 < x < 3$,

(iv) $f(x) = f(-x)$ for $-3 \leq x \leq 3$,

(v) $f(x)$ is concave up and decreasing for $4 < x < 6$.

Solution:



6. [10 points] All problems below are independent of each other.

a. [3 points] Let $m(x) = (1+x^2)^{3x-4}$. Circle the limit below that represents $m'(2)$. There is only one correct answer.

Solution:

(A) $\displaystyle\lim_{h\to 0} \frac{(1+x^2)^{3x-4}+h-25}{h}$

(B) $\displaystyle\lim_{h\to 0} \frac{(1+h^2)^{3h-4}-25}{h}$

(C) $\displaystyle\lim_{h\to 0} \frac{(1+(2+h)^2)^{3h-4}-25}{h}$

(D) $\boxed{\displaystyle\lim_{h\to 0} \frac{(1+(2+h)^2)^{3h+2}-25}{h}}$

(E) $\displaystyle\lim_{h\to 0} \frac{(5+h^2)^{3h+2}-25}{h}$

(F) $\displaystyle\lim_{h\to 2} \frac{(1+h^2)^{3h+2}-25}{h}$

b. [4 points] Let $p(x)$ be a polynomial satisfying all the following properties:

(i) $p(x) = 0$ only at $x = -2, 0, 3$.

(ii) $\displaystyle\lim_{x\to -\infty} p(x) = -\infty$ and $\displaystyle\lim_{x\to\infty} p(x) = -\infty$.

Find one possible formula for $p(x)$. There may be more than one correct answer.

Solution: $p(x) = -x^2(x+2)(x-3)$

c. [3 points] Let $h(x)$ be a rational function satisfying all the following properties:

(i) $\displaystyle\lim_{x\to 2} h(x) = 0$ and h is not defined at $x = 2$.

(ii) $\displaystyle\lim_{x\to\infty} h(x) = 0$.

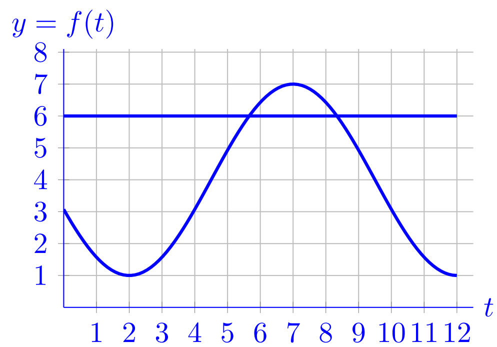
Find one possible formula for $h(x)$. There may be more than one correct answer.

Solution: $h(x) = \dfrac{(x-2)^2}{(x^2+1)(x-2)}$

7. [10 points] An apple farmer wants to assess the damage done by a plague to the trees in his orchard. In order to do so, he installs cameras on a couple of small flying robots to film the damage done by the plague to the trees. Let $f(t)$ and $s(t)$ and be the height above the ground (in feet) of the first and second robot t seconds after they started recording.

a. [5 points] Let $f(t) = 4 - 3\cos\left(\frac{\pi}{5}t - \frac{2\pi}{5}\right)$. Find the time(s) at which the first robot is 6 feet above the ground for $0 \le t \le 12$. Your answer(s) should be *exact*. Show all your work.

Solution: From the graph:



we see that there are two solutions.

$$4 - 3\cos\left(\frac{\pi}{5}t - \frac{2\pi}{5}\right) = 6$$

$$\cos\left(\frac{\pi}{5}t - \frac{2\pi}{5}\right) = -\frac{2}{3}$$

$$\frac{\pi}{5}t - \frac{2\pi}{5} = \cos^{-1}\left(-\frac{2}{3}\right)$$

$$t = \frac{5}{\pi}\left(\cos^{-1}\left(-\frac{2}{3}\right) + \frac{2\pi}{5}\right)$$

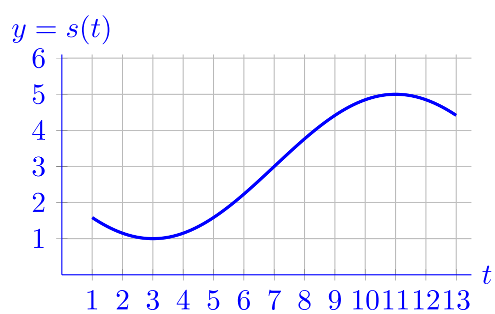
$$\frac{\pi}{5}t - \frac{2\pi}{5} = 2\pi - \cos^{-1}\left(-\frac{2}{3}\right)$$

$$t = 2 + \frac{5}{\pi}\left(2\pi - \cos^{-1}\left(-\frac{2}{3}\right)\right)$$

$$t = \frac{5}{\pi}\left(\cos^{-1}\left(-\frac{2}{3}\right) + \frac{2\pi}{5}\right), 2 + \frac{5}{\pi}\left(2\pi - \cos^{-1}\left(-\frac{2}{3}\right)\right)$$

b. [5 points] The graph of the sinusoidal function $s(t)$ is shown below only for $1 \le t \le 13$. Find a formula for $s(t)$.

Solution:



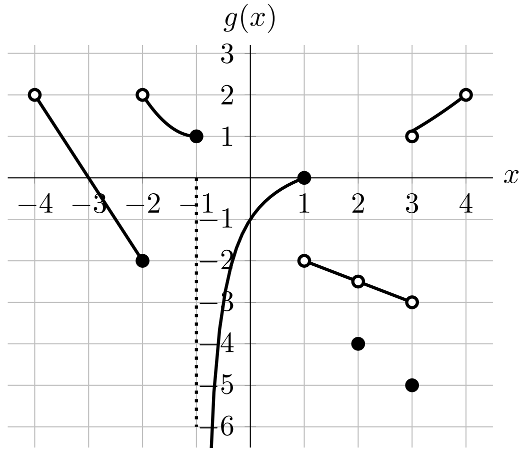
The sinusoidal $s(t) = -A\cos(B(t - h)) + k$ has:

- Amplitude $= 2$ then $A = 2$.
- Midline $y = 3$ then $k = 3$.
- Period $= 16$ then $B = \frac{2\pi}{16} = \frac{\pi}{8}$.
- Horizontal shift $= 3$ then $h = 3$.

Hence $s(t) = -2\cos\left(\frac{\pi}{8}(t - 3)\right) + 3$ (other formulas are also possible).

8. [15 points] Consider the functions $f(x)$ and $g(x)$ given by the formula and graph below.

$$f(x) = \begin{cases} 2x^3 - 2x^2 & \text{for } x \le 1, \\ x^3 + 1 & \text{for } x > 1. \end{cases}$$



a. [5 points] Circle the correct answer(s) in each of the following questions.

Solution:

i) At which of the following values of x is the function $g(x)$ <u>not</u> continuous?

$x = -3$ **[$x = -1$]** $x = 0$ **[$x = 2$]** $x = 3.5$

ii) At which of the following values of x is the function $f(x) + g(x)$ continuous?

$x = -2$ $x = -1$ **[$x = 0$]**   **[$x = 1$]** $x = 2$

Note that $g(x)$ is linear on the interval $(-4, -2)$,$(1, 2)$ and $(2, 3)$. All your answers below should be *exact*. If any of the quantities do not exist, write DNE.

b. [2 points] Find $\lim_{x \to 2} (2f(x) + g(x))$.

Solution: $2(2^3 + 1) - 2.5 = 15.5$ **Answer:** 15.5

c. [2 points] Find $\lim_{x \to \infty} \dfrac{f(2x)}{x^3}$.

Solution: $\lim_{x \to \infty} \dfrac{f(2x)}{x^3} = \lim_{x \to \infty} \dfrac{8x^3 + 1}{x^3} = 8$ **Answer:** 8

d. [2 points] Find $\lim_{x \to \infty} g\left(x^2 e^{-x} + 3\right)$.

Solution: $\lim_{x \to \infty} g\left(x^2 e^{-x} + 3\right) = \lim_{u \to 3^+} g(u) = 1$ **Answer:** 1

e. [2 points] For which value(s) of p does $\lim_{x \to p^+} g(x) = 1$?

Solution: **Answer:** $p = -3.5, 3$

f. [2 points] Find $\lim_{x \to -1^-} f(-x)$.

Solution: **Answer:** 2

9. [12 points] A new video is released and a few hours later it goes viral. The number of views, in thousands, of the video t hours after it goes viral is given by the function $v(t)$. For the first 24 hours, the number of views of the video is increasing exponentially, reaching 50,000 views 12 hours after going viral and 120,000 views 24 hours after going viral. After that, during the second 24 hours, the video is gaining 10,000 views every 3 hours.

a. [8 points] Find a piecewise defined formula for $v(t)$ for $0 \leq t \leq 48$. Show all your work.

Solution: For $0 \leq t \leq 24$, let $v(t) = ab^t$, then $ab^{12} = 50$ and $ab^{24} = 120$. Then, dividing, we see

$$b^{12} = 2.4 \qquad b = (2.4)^{\frac{1}{12}}.$$

$$ab^{12} = 50 \quad \text{yields} \quad a = \frac{50}{2.4}$$

For $24 \leq t \leq 48$, $v(t) = a + m(t-24)$. In this case $v(24) = a = 120$ and $m = \frac{10}{3}$.

$$v(t) = \begin{cases} \frac{50}{2.4}(2.4)^{\frac{t}{12}} & \text{for } 0 \leq t \leq 24 \\ 120 + \frac{10}{3}(t-24) & \text{for } 24 < t \leq 48. \end{cases}$$

b. [2 points] Find the hourly percentage growth rate of $v(t)$ during the first 24 hours. Your answer should be given as a percentage accurate up to the first two decimals.

Solution: $r = (2.4)^{\frac{1}{12}} - 1 \approx 0.07568$ then 7.57 %.

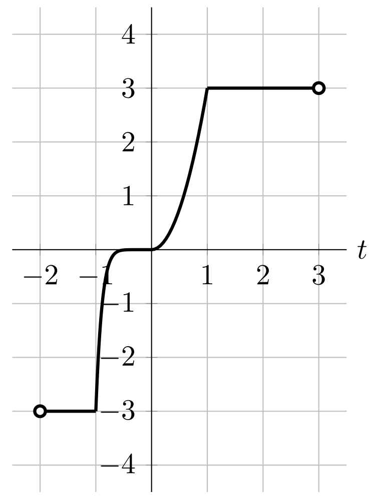
c. [2 points] During the third day, the number of views of the video is not given by a nice formula, but it is at least known that $v'(53) = 6$. What are the units of 6?

Solution: thousands of views per hour.

10. [6 points] The graph of $r(t)$ with domain $(-2, 3)$ is given below.

In each of the following parts, the graph of a function obtained from r by one or more transformations is shown. *Note that the graphs are not all drawn at the same scale.*

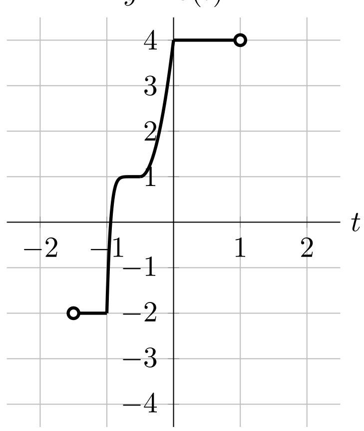
$y = r(t)$



Find a formula for $s(t)$ and $u(t)$ in terms of the function r. You do not need to show work in this problem.

a. [3 points]

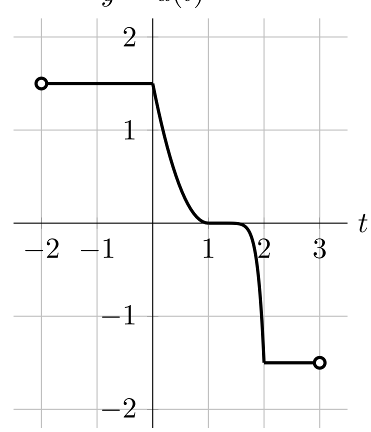
$y = s(t)$



Solution: $s(t) = r(2(t + 0.5)) + 1$

b. [3 points]

$y = u(t)$



Solution: $u(t) = \frac{1}{2} r(-(t - 1))$