Math 115 — Second Midterm — March 20, 2018.

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	12	
3	12	
4	12	
5	15	
6	5	

Problem	Points	Score
7	10	
8	14	
9	7	
Total	100	

1. [13 points] Some values of the twice differentiable function $f(x)$ and of its first and second derivative are given by the following table

x	0	1	2	4	5	6	7
$f(x)$	1			4	4.3	5	
$f'(x)$			8		0.25	0.6	2
$f''(x)$	4				0.1	0.2	

Suppose the function $f(x)$ is defined and invertible for $-\infty < x < \infty$. In the following questions, you will find some of the missing values using the information given. If there is not enough information given to answer the question, write "NEI". Show your work.

a. [4 points] The function $a(x) = \ln(1 + f(x))$ satisfies $a'(2) = 2$. Find $f(2)$.

Solution:

$$\begin{aligned} a'(x) &= \frac{1}{1+f(x)} f'(x) \\ 2 &= \frac{8}{1+f(2)} \\ 8 &= 2 + 2f(2) \\ f(2) &= 3 \end{aligned}$$

Answer: $f(2) = 3.$

b. [3 points] Let $b(x) = f(x)f'(x)$ and $b'(0) = 4$. Find $f'(0)$.

Solution:

$$\begin{aligned} b'(x) &= (f'(x))^2 + f(x)f''(x) \\ 4 &= (f'(0))^2 + f(0)f''(0) \\ 4 &= (f'(0))^2 + 4 \\ f'(0) &= 0. \end{aligned}$$

Answers: $f'(0) = 0.$

c. [3 points] The quadratic approximation $Q(x)$ of the function $f(x)$ at $x = 1$ is

$Q(x) = \frac{1}{2}x + \frac{3}{2}$. Find $f(1)$, $f'(1)$, and $f''(1)$.

Solution:

Answers: $f(1) = 2$, $f'(1) = \frac{1}{2}$, $f''(1) = 0$

d. [3 points] Let $h(x) = f^{-1}(x)$. Find the value of $h'(5)$.

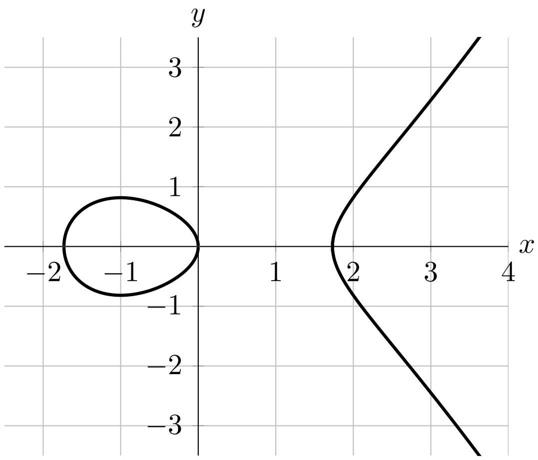
Solution: $h'(x) = \frac{1}{f'(f^{-1}(x))}$, then $h'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(6)} = \dfrac{1}{0.6} = \dfrac{5}{3}$.

Answer: $h'(5) = \dfrac{5}{3}$

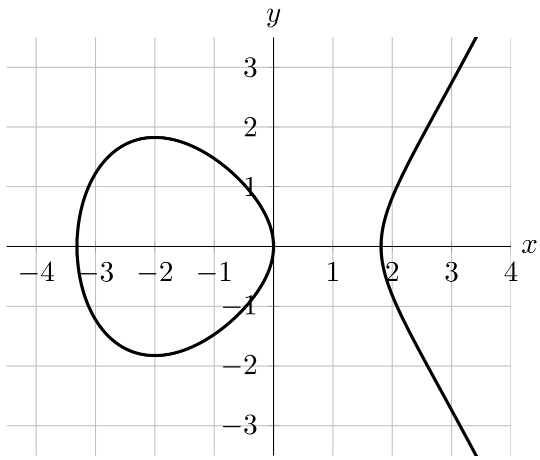
2. [12 points]

a. [6 points] Each the following is the graph of an implicit function.

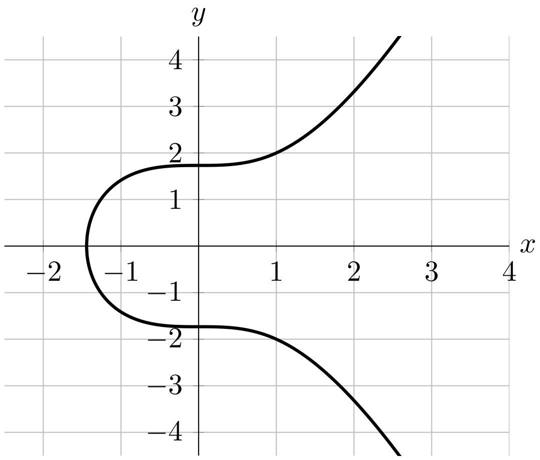
Graph I



Graph II



Graph III



Match each of the graphs above to the formula below that gives the slope at each point on the graph.

(A) $\dfrac{dy}{dx} = \dfrac{3x^2}{2y}$,

(B) $\dfrac{dy}{dx} = \dfrac{(x-1)(x+2)}{2y}$,

(C) $\dfrac{dy}{dx} = \dfrac{x^2-1}{2y}$,

(D) $\dfrac{dy}{dx} = \dfrac{(y-1)(y+2)}{2x}$.

You do not need to show work in this part.

Solution:

Answers: Graph I: **C** , Graph II: **B** , Graph III: **A**

b. [6 points] Find $\frac{dy}{dx}$ for the implicit function given by

$$2^{x+y} + \sin(x)\cos(y) = 5 - x.$$

Show all your work carefully to receive full credit.

Solution:

$$\frac{d}{dx}\left(2^{x+y} + \sin(x)\cos(y)\right) = \frac{d}{dx}(5 - x)$$

$$\ln(2)2^{x+y}\left(1 + \mathbf{\frac{dy}{dx}}\right) + \left(-\sin(x)\sin(y)\mathbf{\frac{dy}{dx}} + \cos(x)\cos(y)\right) = -1$$

$$\left(\ln(2)2^{x+y} - \sin(x)\sin(y)\right)\mathbf{\frac{dy}{dx}} + \ln(2)2^{x+y} + \cos(x)\cos(y) = -1$$

$$\left(\ln(2)2^{x+y} - \sin(x)\sin(y)\right)\mathbf{\frac{dy}{dx}} = -(1 + \ln(2)2^{x+y} + \cos(x)\cos(y))$$

Answer: $\frac{dy}{dx} = -\frac{1 + \ln(2)2^{x+y} + \cos(x)\cos(y)}{\ln(2)2^{x+y} - \sin(x)\sin(y)}$

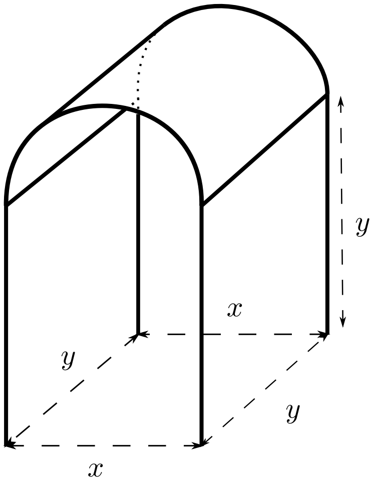
3. [12 points] The Public Transit Authorities (PTA) are designing rain shelters for their bus stops. They decide to place a roof in the shape of half a cylinder on four vertical legs of height y feet. The four legs are placed in a *rectangle* on the ground with width x feet and length y feet.

The costs of production are:

- \$25 for each foot of the total length of the legs,
- \$40 for each square foot of the area of the roof.

The following formulas may be useful in this problem:

- the surface area of a cylinder of radius r and length ℓ is $2\pi r\ell$,
- the volume of a cylinder of radius r and length ℓ is $\pi r^2\ell$.



The PTA would like to spend exactly \$5000 on one rain shelter.

a. [5 points] Find a formula for y in terms of x.

Solution: We have that

$$25\cdot(4y) + 40\cdot\frac{1}{2}\left(2\pi\left(\frac{x}{2}\right)\right)y = 5000$$

so

$$y(100+20\pi x) = 5000,$$

$$y = \frac{5000}{100+20\pi x}.$$

Answer: $y = \dfrac{250}{5+\pi x}$.

b. [4 points] Find a formula for the total volume in cubic feet covered by the shelter, $V(x)$, if the width of the dashed rectangle has length x feet.

Solution: The volume is

$$V = xy^2 + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 y,$$

and hence **Answer:** $V(x) = x\cdot\left(\frac{250}{5+\pi x}\right)^2 + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2\cdot\left(\frac{250}{5+\pi x}\right)$.

c. [3 points] The PTA wants to make sure that *each* of the sides of the rectangle has length at least 5 feet, and the height (that is, y) of the shelter is at least 8 feet. In the context of the problem, what is the domain of the function $V(x)$?

Solution: We know that $x \geq 5$ and also $y \geq 8$. Therefore, $\dfrac{250}{5+\pi x} = y \geq 8$ $\quad 250 \geq 40 + 8\pi x \quad x \leq \dfrac{210}{8\pi} = \dfrac{105}{4\pi}$.

Answer: $5 \leq x \leq \dfrac{105}{4\pi}$

4. [12 points] In the following questions, use calculus to justify your answers and show enough evidence to demonstrate that you have found them all. Determine your answers algebraically.

a. [7 points] Let $f(x)$ be a continuous function defined for all real numbers with derivative given by

$$f'(x) = \frac{(2x+1)(x-2)^2}{(x+3)^{\frac{1}{3}}}.$$

Find the x-coordinate(s) of the local maximum(s) and local minimum(s) of the function $f(x)$. Write "NONE" if the function has no local maximum(s) and/or local minimum(s).

Solution: The critical points of $f(x)$ are $x = -3$, $-\frac{1}{2}$, and 2.
To classify them, note that

- for $x < -3$, $f'(x) = \frac{(-)(+)}{(-)} = (+)$,
- for $-3 < x < -\frac{1}{2}$, $f'(x) = \frac{(-)(+)}{(+)} = (-)$,
- for $-\frac{1}{2} < x < 2$, $f'(x) = \frac{(+)(+)}{(+)} = (+)$,
- for $x > 2$, $f'(x) = \frac{(+)(+)}{(+)} = (+)$.

Therefore, $f(x)$ has a local maximum at $x = -3$ and a local minimum at $x = -\frac{1}{2}$.

Answers: Local maximum(s) at $x = -3$ Local minimum(s) at $x = -\frac{1}{2}$

b. [5 points] Let $g(x)$ be a continuous function defined for all real numbers with second derivative given by

$$g''(x) = (2^x - 4)(x^2 - 4).$$

Find the x-coordinates of the inflection points of the function $g(x)$. Write "NONE" if the function has no inflection points.

Solution: Note that $g''(x)$ is defined everywhere, and $g''(x) = 0$ for $x = -2$ and $x = 2$. We check for a change of sign in $g''(x)$ at these points to see if they are inflection points:

- for $x < -2$, $g''(x) = (-)(+) = (-)$,
- for $-2 < x < 2$, $g''(x) = (-)(-) = (+)$,
- for $x > 2$, $g''(x) = (+)(+) = (+)$.

Therefore, $x = -2$ is the only inflection point of $g(x)$.

Answer: Inflection point(s) at $x = -2$

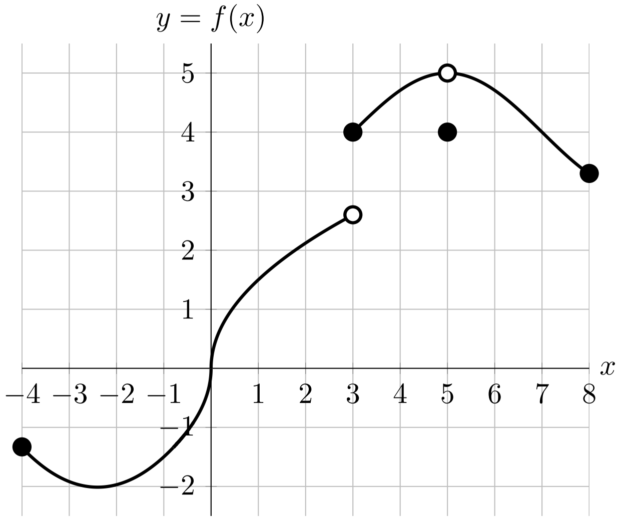
5. [15 points] The graph of the function $f(x)$ with domain $-4 \leq x \leq 8$ is shown below. The function $f(x)$ satisifies:

- $f(x) = 1.5x^{\frac{1}{3}}$

  for $-1 < x < 1$,

- $f(x) = 4 + \sin\left(\frac{\pi}{4}(x-3)\right)$

  for $3 \leq x < 5$ and $5 < x \leq 8$.



a. [2 points] Estimate the x-coordinate(s) of all the local minimum(s) of $f(x)$ in $-4 < x < 8$. Write "None" if $f(x)$ does not have any local minimums.

Solution:

Answer: $x = -2.4, 5$

b. [3 points] Find the value(s) of b in $-4 < b < 8$ for which the limit $\lim\limits_{h \to 0} \dfrac{f(b+h) - f(b)}{h}$ does *not exist*. Write "None" if there are no such values of b.

Solution:

Answer: $b = 0, 3, 5$

c. [4 points] Estimate the x-coordinate(s) of all critical points of $f(x)$ in $-4 < x < 8$. Write "None" if $f(x)$ does not have any critical points.

Solution:

Answer: $x = -2.4, 0, 3, 5$

d. [3 points] On which of the following intervals is the *conclusion* of the Mean Value Theorem true? Circle your answer.

Solution:

$\boxed{[-4, 0]}$ $\boxed{[0, 5]}$ $[1, 3]$ $[3, 7]$ None

e. [3 points] On which of the following intervals are the *hypotheses* of the Mean Value Theorem true? Circle your answer.

Solution:

$\boxed{[-3, -1]}$ $[-2, 2]$ $\boxed{[0, 2]}$ $[3, 5]$ None

6. [5 points] The function $P(t)$ is given by the equation

$$P(t) = \begin{cases} t+4 & t<2 \\ t^2-3t+8 & 2 \leq t \leq 3 \\ \frac{1}{9}(t^3+44) & t>3 \end{cases}$$

For which values of t is $P(t)$ *differentiable*? Show all your work to justify your answer.

Solution:

- At $t=2$:
  - Continuity:

$$\lim_{t\to 2^-} t+4=6 \quad \text{and} \quad \lim_{t\to 2^+} t^2-3t+8=6 \quad \text{and} \quad P(2)=6.$$

    Hence $P(t)$ is continuous at $t=2$.
  - Differentiability: The function $g(t)=t+4$ satisfies $g'(t)=1$ and hence $g'(2)=1$. Similarly, it $h(t)=t^2-3t+8$ satisfies $h'(t)=2t-3$ and $h'(2)=1$. So

$$\lim_{t\to 2-} P'(t)=1=\lim_{t\to 2+} P'(t).$$

  Hence $P(t)$ is differentiable at $t=2$ (with $P'(2)=1$).
- At $t=3$:
  - Continuity:

$$\lim_{t\to 3^-} t^2-3t+8=8 \quad \text{and} \quad \lim_{t\to 3^+} \frac{1}{9}(t^3+44)=\frac{71}{9}\neq 8.$$

    Hence $P(t)$ is not continuous at $t=3$ and therefore not differentiable at $t=3$.
- All the functions (polynomials) involved in the formula of $P(t)$ are differentiable on the domains assigned to them.

Hence $P(t)$ is differentiable for all $t\neq 3$.

Answer: The function $P(t)$ is differentiable for the following values of t: all $t\neq 3$.

7. [10 points] The amount of chlorine in a chemical reaction $C(t)$ (in gallons) t seconds after it has been added into a solution is given by the function

$$C(t) = 2 - 3(t-5)^{\frac{4}{5}}(t-1)e^{-t} \qquad \text{for} \qquad t \geq 0.$$

Notice that

$$C'(t) = \frac{3(t-6)(5t-9)e^{-t}}{5(t-5)^{1/5}}.$$

a. [8 points] Use calculus to find the time(s) (if any) at which the amount of chlorine in the solution is the greatest and the smallest. If the function has no global maximum or global minimum write "None" in the appropriate space. Show all your work.

Solution: Critical points:

- $C'(t) = 0$: $t = 6$ and $t = 1.8$.
- $C'(t)$ undefined: $t = 5$.

Finding the output values of $C(t)$ at critical points and the behavior of the function at the endpoints:

t	0	1.8	5	6
$C(t)$	$2 + 3(5)^{\frac{4}{5}} \approx 12.87$	≈ 0.994	2	≈ 1.96

$$\lim_{t\to\infty} 2 - 3(t-5)^{\frac{4}{5}}(t-1)e^{-t} = 2.$$

Answer: Global maximum(s) at $t = 0$

Global minimum(s) at $t = 1.8$.

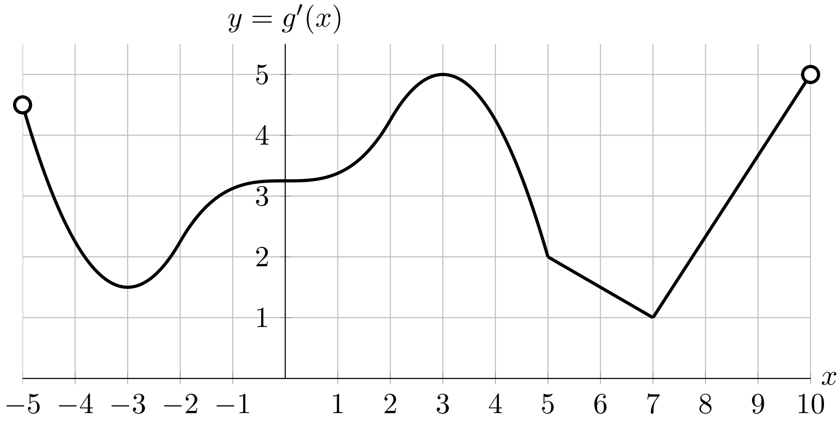
b. [2 points] What is the maximum amount of chlorine in the solution? If there is never a maximum amount of chlorine in the solution, write "None".

Solution: **Answer:** $2 + 3(5)^{\frac{4}{5}} \approx 12.87$ gallons.

8. [14 points] The graph of the **derivative** $g'(x)$ of the function $g(x)$ with domain $-5 < x < 10$ is shown below.

The function $g'(x)$ has corners at $x = 5$ and $x = 7$, and it is linear on the intervals $(5, 7)$ and $(7, 10)$.

If there is not enough information given to answer the question, write "NEI". If the answer is none, write "NONE".



a. [3 points] Estimate the interval(s) on which the function $g(x)$ is concave up.

Solution:

Answer: (-3,3) and (7,10)

b. [3 points] Estimate all the x-coordinates of the inflection points of $g(x)$.

Solution:

Answer: $x = -3, 3, 7.$

c. [2 points] Estimate the values of x in $-5 < x < 10$ for which $g''(x)$ is not defined.

Solution:

Answer: $x = 5, 7.$

d. [2 points] Estimate the interval(s) on which $g'''(x) > 0$. Recall that $g'''(x)$ is the derivative of $g''(x)$.

Solution:

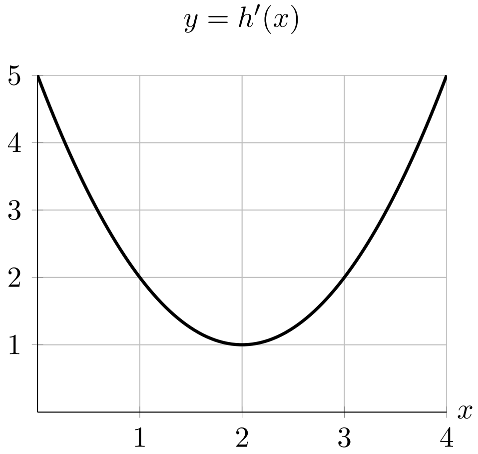
Answer: (approximately) $(-5, -2)$ and $(0, 1.8)$.

e. [4 points] Let $P(x)$ be the quadratic approximation of $g(x)$ at $x = 8$. Find the formula of $P(x)$ in terms of only the variable x if $g(8) = -2$. Your answer should not include the letter g.

Solution: $g(8) = -2$, $g'(8) = 1 + \frac{4}{3} = \frac{7}{3}$ and $g''(8) = \frac{4}{3}$. Then

Answer: $P(x) = -2 + \dfrac{7}{3}(x - 8) + \dfrac{2}{3}(x - 8)^2$

9. [7 points] The graph of $h'(x)$ (the **derivative** of $h(x)$) is shown below.



a. [3 points] Find a formula for the tangent line approximation $L(x)$ to the function $h(x)$ near $x = 2$ if the point $(2, -3)$ lies on the graph of $y = h(x)$. Your answer should not include the letter h.

Solution: $h(2) = -3$ and $h'(2) = 1$.

Answer: $L(x) = -3 + (x - 2)$

b. [1 point] Use the tangent line approximation to the function $h(x)$ near $x = 2$ to approximate the value of $h(1.6)$.

Solution:

Answer: $h(1.6)$ is approximately $L(1.6) = -3 + (1.6 - 2) = -3.4$.

c. [3 points] Is your approximation in part **b** an overestimate or an underestimate? Circle your answer and give a justification of your answer.

Solution:

[✓ Overestimate] Underestimate Not enough information

Justification:

Since $h'(x)$ is decreasing on $[1.6, 2]$, $h(x)$ is concave down on $[1.6, 2]$. Hence the approximation is an overestimate.