Math 115 — Final Exam — April 19, 2018

EXAM SOLUTIONS

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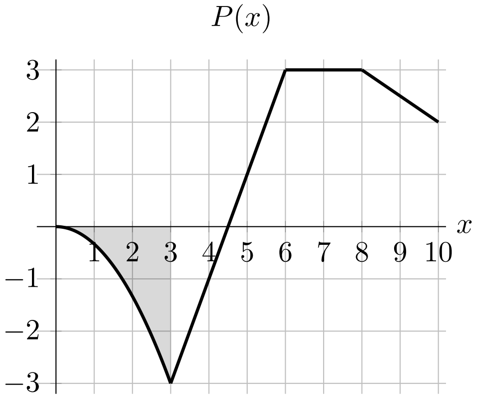
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 14 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	15	
3	16	
4	8	
5	6	
6	14	

Problem	Points	Score
7	11	
8	6	
9	11	
Total	100	

1. [13 points] The function $P(x)$ is defined on the interval $-14 \le x \le 14$. The graph of $P(x)$ is shown below for $0 \le x \le 10$.



The function $P(x)$ has the following properties:

- it is an even function,
- the shaded region has area equal to 3,
- $P(x)$ is twice differentiable on $(9, 14)$ and P, P', and P'' have the following values

x	10	11	12	13
$P(x)$	2	2.5	3	4
$P'(x)$	-0.5	0.2	-2	1.5
$P''(x)$	0	-0.5	1.7	2.5

In the following questions, your answers must be **exact**. If any of the answers are undefined, write "UND". If there is not enough information to answer a question, write "NEI".

a. [2 points] Find $\lim\limits_{m \to 0} \dfrac{P(m+12) - P(12)}{m}$.

Solution: $P'(12) = -2$

Answer: -2

b. [2 points] Let $J(x)$ be an antiderivative of $P(x)$. Find $J'(3)$.

Solution: $J'(3) = P(3) = -3.$

Answer: $-3.$

c. [2 points] Let $K(x)$ be an antiderivative of $P(x)$ with $K(8) = -2$. Find $K(0)$.

Solution: $K(8) - K(0) = \int_0^8 P(x)dx$ so $K(0) = -2 - 3 = -5$

Answer: $-5.$

d. [3 points] Find $\int_{-3}^{6} (2P(t) + 1)dt$.

Solution: $\int_{-3}^{6} (2P(t) + 1)dt = 2\int_{-3}^{6} P(t)dt + 9 = 2(-6) + 9 = -3$

Answer: $-3.$

e. [2 points] Find $\int_{10}^{13} P''(x)dx$.

Solution: $\int_{10}^{13} P''(x)dx = P'(13) - P'(10) = 1.5 - (-0.5) = 2$

Answer: 2.

f. [2 points] Let $Q(x) = P(3x^2 + 1)$. Find $Q'(2)$.

Solution: $Q'(x) = P'(3x^2 + 1)(6x)$ then $Q'(2) = P'(13)(12) = (1.5)(12) = 18$

Answer: 18.

2. [15 points] There were 3 trillion trees in the world in the year 2000.

- Since the year 2000, a group of environmentalists have recorded the number of trees lost in the world due to natural causes or due to human activities. Let $C(t)$ be the rate at which the number of trees decreases due to any of these causes, t years after the year 2000, in trillions of trees per year.
- At the same time, some governments and other organizations plant new trees to increase the number of trees in the world. The group is also measuring the rate $P(t)$ at which the trees are being planted, t years after the year 2000, in trillions of trees per year.

Throughout this question, you may assume that the functions $C(t)$ and $P(t)$ describe the only changes to the number of trees in the world.

a. [7 points] In parts (i) and (ii) below, give a mathematical expression that may involve $C(t)$, $P(t)$, their derivatives, and/or definite integrals.

(i) Find an expression for the total number of trees in the world (in trillions) in the year 2005.

Solution: **Answer:** $3 + \int_0^5 (P(t) - C(t))dt$

(ii) Find an expression for the average rate at which the trees were being planted (in trillions of trees per year) between the years 2002 and 2009.

Solution:

Answer: $\frac{1}{7}\int_2^9 P(t)dt$

b. [3 points] Write a practical interpretation of the statement $\int_{13}^{17} C(t)dt = 0.05$. Your answer must be a complete sentence.

Solution: Between 2013 and 2017, 0.05 trillion (50 billion) trees were cut.

The question has been reproduced here for your convenience.

There were 3 trillion trees in the world in the year 2000.

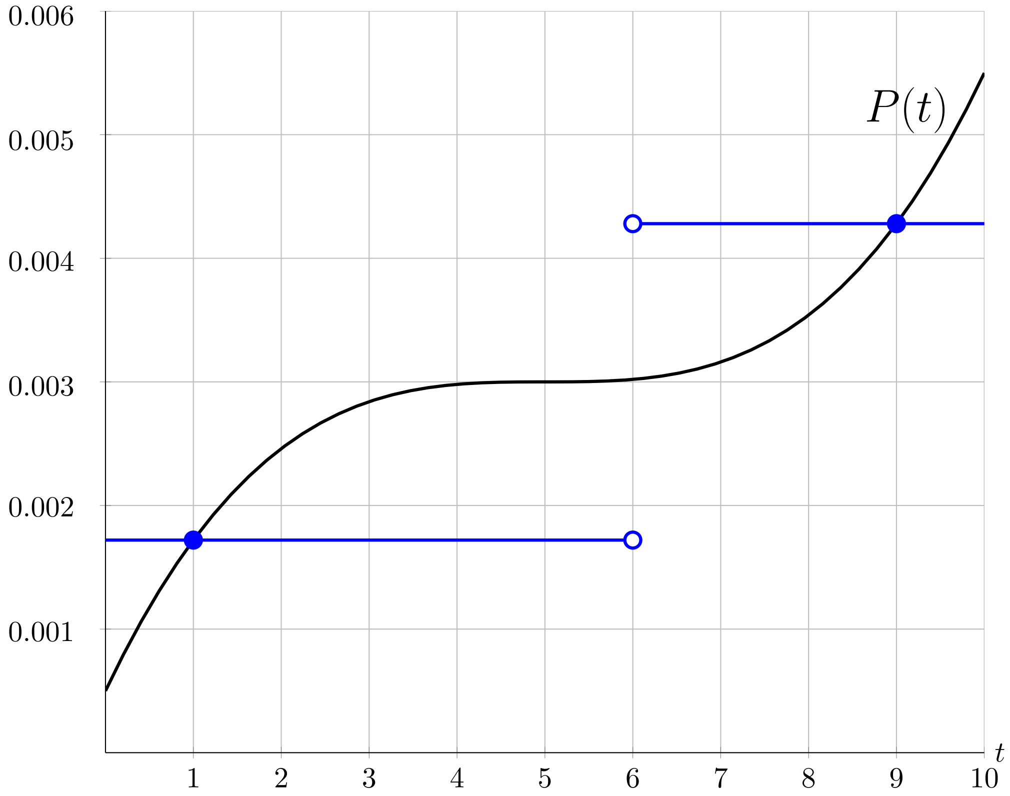
- Since the year 2000, a group of environmentalists have recorded the number of trees lost in the world due to natural causes or due to human activities. Let $C(t)$ be the rate at which the number of trees decreases due to any of these causes, t years after the year 2000, in trillions of trees per year.
- At the same time, some governments and other organizations plant new trees to restore the forests. The group is also measuring the rate $P(t)$ at which the trees are being planted, t years after the year 2000, in trillions of trees per year.

Throughout this question, you may assume that the functions $C(t)$ and $P(t)$ describe the only changes to the number of trees in the world.

c. [5 points] Additionally, you know that

- $C(t) = P(t)$ in 2001 and 2009,
- between 2000 and 2010, the number of trees in the world was the largest in 2006.

The graph of $P(t)$ is given below for $0 \leq t \leq 10$. In the same axis, **sketch a possible graph** of $C(t)$ that is consistent with the above information. Note that there may be many correct answers.



3. [16 points] Last summer, Brad and Angelina set up a lemonade stand where they sold lemonade charging 60 cents per ounce. In other words, $MR(q) = 0.6$ (in dollars per ounce) where q is the number of ounces of lemonade they sold.

a. [2 points] Find a formula for their revenue $R(q)$ (in dollars) were q is the amount (in ounces) of lemonade sold. Assume their initial revenue is zero dollars.

Solution: **Answer:** $R(q) = 0.6q$.

Since they are using utensils they already have, they have no fixed costs. They can produce at most 120 ounces of lemonade, and the **marginal cost** function, $MC(q)$, is:

- continuous for $0 < q < 120$,
- concave down for $0 < q < 120$,
- increasing for $0 < q < 60$ and decreasing for $60 < q < 120$.

Brad and Angelina recorded some of the values of $MC(q)$ (in dollars per ounce) in the following table:

q	0	15	30	45	60	75	90	105	120
$MC(q)$	0.15	0.45	0.6	0.7	0.75	0.7	0.6	0.45	0.15

b. [3 points] Recall that $C(60) - C(0) = \int_0^{60} MC(q)\, dq$. Estimate $C(60)$ by using a right-hand Riemann sum with 2 equal subdivisions. Make sure to write down all terms in your sum.

Solution: **Answer:** $30(0.6 + 0.75) = 40.5$

c. [1 point] Is your estimate in **b** an overestimate or an underestimate of $C(60)$? Circle your answer.

Solution:

[✓ Overestimate] Underestimate Not Enough Information

d. [3 points] Suppose Brad and Angelina want to use a Riemann sum to calculate $C(60)$, accurate within 50 cents of the actual value. At least how many times, using equal intervals on $[0, 60]$, should Brad and Angelina have measured $MC(q)$ in order to guarantee this accuracy? Justify your answer.

Solution:

$$\begin{aligned}\text{Difference between righthand and lefthand sums} &= (MC(60) - MC(0))\Delta q \\ &= (0.75 - 0.15)\Delta q = 0.6\Delta q \leq 0.5.\end{aligned}$$

yields $\Delta q \leq \frac{5}{6} \approx .833$. Recall that $\Delta q = \frac{b-a}{n}$ where n is the number of measurements. This gives:

$$n = \frac{60}{\Delta q} = \frac{60}{5/6} = 72.$$

Answer: They should have measured it at least 72 times.

A part of the question has been reproduced for your convenience.

Brad and Angelina have no fixed costs and they can produce at most 120 ounces of lemonade. The **marginal cost** function (in dollars per ounce), $MC(q)$, is:

- continuous for $0 < q < 120$,
- concave down for $0 < q < 120$,
- increasing for $0 < q < 60$ and decreasing for $60 < q < 120$.

and some of its values are:

q	0	15	30	45	60	75	90	105	120
$MC(q)$	0.15	0.45	0.6	0.7	0.75	0.7	0.6	0.45	0.15

The marginal revenue is $MR(q) = 0.6$ (in dollars per ounce).

e. [2 points] Find all the critical points of the profit function $\pi(q)$.

Solution: **Answer:** $q = 30, 90$.

The table below gives some values of $C(q)$ (in dollars):

q	15	30	45	60	75	90	105	120
$C(q)$	4.7	12.7	22.5	33.5	44.5	54.3	62.8	67

Use the values in the table to answer the following question.

f. [5 points] How many ounces of lemonade should Brad and Angelina sell in order to maximize their profit, and what is their maximal profit? Use calculus to fully justify that your answer is a global maximum. Remember to include units in your answer.

Solution: The critical points are $q = 30, 90$ and the end points are $q = 0$ and $q = 120$.

q	0	30	90	120
$R(q)$	0	18	54	72
$C(q)$	0	12.7	54.3	67
$\pi(q) = R(q) - C(q)$	0	5.3	-0.3	5

Answer: Brad and Angelina's profit is maximized when they sell 30 ounces

of lemonade, and their maximal profit is 5.30 dollars.

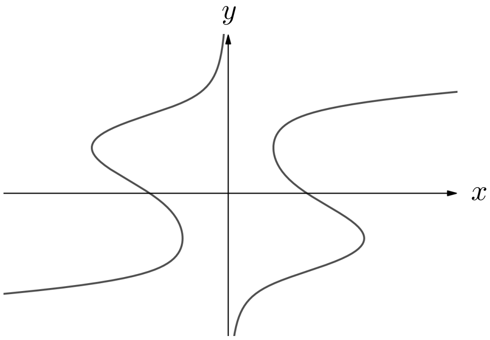
4. [8 points] Consider the curve $\mathcal{C}$ given by

$$3xy + x^2 = xy^3 + 3,$$

and note that it satisfies

$$\frac{dy}{dx} = \frac{2x - y^3 + 3y}{3xy^2 - 3x}.$$

The graph of $\mathcal{C}$ is shown below



a. [3 points] Find an equation of the tangent line to the curve $\mathcal{C}$ at the point $(3, 2)$.

Solution:

$$m = \frac{2(3) - (2)^3 + 3(2)}{3(3)(2)^2 - 3(3)} = \frac{4}{27}.$$

Answer: $y = 2 + \frac{4}{27}(x - 3)$

b. [5 points] Find the coordinates (x, y) of the point(s) with $y > 0$ at which the curve $\mathcal{C}$ has a vertical tangent line. Show all your work to justify your answer(s).

Solution: We need to solve

$$3xy^2 - 3x = 0$$
$$3x(y - 1)(y + 1) = 0.$$

yields $x = 0$ or $y = \pm 1$.
In the case of $x = 0$, the y-coordinates of the points $(0, y)$ in the curve $\mathcal{C}$ satisfy

$$3(0)y + (0)^2 = (0)y^3 + 3$$

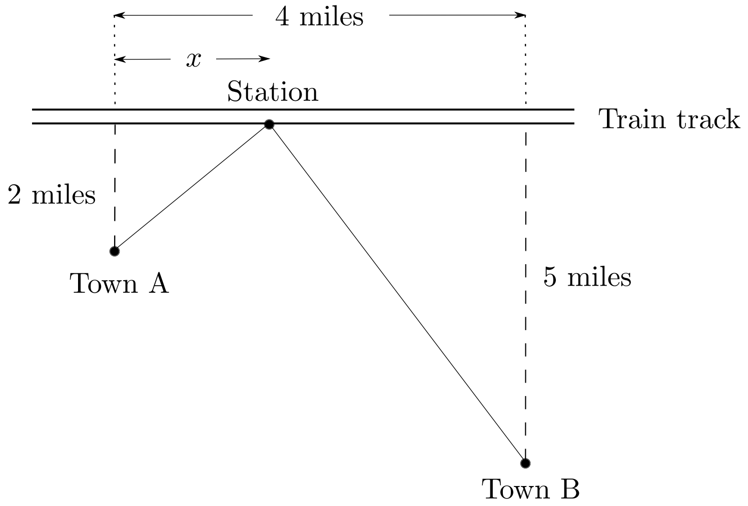
which yields to $0 = 3$. That means that there are no points in the curve $\mathcal{C}$ with x-coordinate equal to 0.
Since we are looking for points above the x-axis ($y > 0$), then we only look for points $(x, 1)$ in the curve $\mathcal{C}$. Then the x-coordinates satisfy

$$3x(1) + x^2 = x(1)^3 + 3$$
$$x^2 - 2x - 3 = 0$$
$$(x - 1)(x + 3) = 0 \quad x = 1, -3.$$

Answer: The coordinates are $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{-3}, \mathbf{1})$.

5. [6 points] There are two towns A and B that are 2 and 5 miles away from a train track and town B is 4 miles to the right of town A (see picture below). The railway company is considering building a train station in between the two towns. The new station will be built next to the train tracks and the local authorities have agreed to build roads directly connecting the two towns to the station.

The picture below shows a map of the towns and the train track along with the distances and a potential location of the station [the picture is not to scale]. Note that x denotes the distance (in miles) from the station to the point closest to Town A from the train track.



a. [3 points] Find a formula for $D(x)$, the sum of the distances (in miles) from Town A to the station and Town B to the station if the station is x miles along the track to the right of Town A. **Circle the best answer.**

Solution:

(i) $D(x) = \sqrt{2 + x^2} + \sqrt{5 + (4 - x)^2}$

(iv) $\boxed{D(x) = \sqrt{2^2 + x^2} + \sqrt{5^2 + (4 - x)^2}}$

(ii) $D(x) = \sqrt{2^2 + x^2} + \sqrt{5^2 + x^2}$

(v) $D(x) = \sqrt{2 + x} + \sqrt{5 + (4 - x)}$

(iii) $D(x) = \sqrt{2^2 + (4 - x)^2} + \sqrt{5^2 + x^2}$

(vi) $D(x) = \sqrt{4^2 + x^2} + \sqrt{2^2 + (5 - x)^2}$

b. [3 points] The people who live in Town A negotiated a deal with the railway company that guarantees the station will be within 3 miles of their town. The railway company will build the station in between the two towns (to the right of Town A and to the left of Town B). Given this information, what is the domain for of $D(x)$?

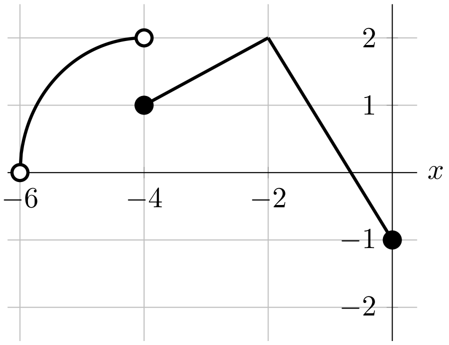
Solution: The distance from Town A to the station at a distance x miles on the side of the tracks is $\sqrt{4+x^2}$. Then we need

$$\begin{aligned} \sqrt{4+x^2} &\leq 3 \\ x^2 &\leq 5 \qquad -\sqrt{5} \leq x \leq \sqrt{5}. \end{aligned}$$

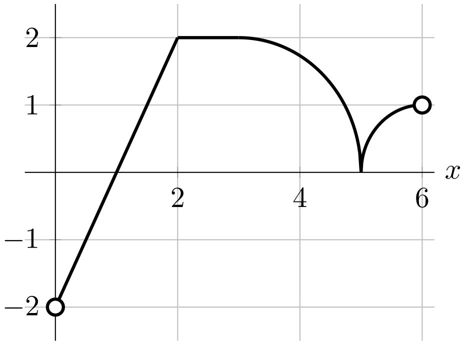
Since the station is between the two towns on the side to the tracks, x cannot be negative. The domain is therefore: $0 \leq x \leq \sqrt{5}$.

6. [14 points] The function $f(x)$ is defined on the interval $-6 < x < 6$. The graphs of $f(x)$ and its derivative $f'(x)$ are shown below on the intervals $(-6, 0]$ and $(0, 6)$ respectively. All the graphs consist of line segments and quarters of circles.

$y = f(x)$



$y = f'(x)$



The function $f(x)$ is continuous at $x = 0$. In the following questions, your answers must be **exact**. If any of the answers are undefined write "UND". If there is not enough information to answer a question, write "NEI"

a. [2 points] Find $\lim_{x\to 4^+}(5f(-x)+3)$.

Solution: **Answer:** 13

b. [2 points] Find $\lim_{x\to -\infty} f(-4-2^x)$.

Solution: **Answer:** 2

c. [2 points] On which interval(s) in $-6 < x < 6$ is the function $f(x)$ is decreasing?

Solution: **Answer:** $[-2, 1]$.

d. [3 points] At which value(s) of $-6 < x < 6$ is the function **not** differentiable?

Solution: **Answer:** $x = -4, -2, 0$.

e. [3 points] Find the coordinates (x, y) of the global maximum of $f(x)$ for $0 \le x \le 5$. Show your work.

Solution: Global maximum at $x = 5$ and its y-coordinate is equal to

$$f(5) = -1 + \int_0^5 f'(x)dx = -1 + 2 + \frac{1}{4}\pi(2)^2 = 1 + \pi$$

Answer: $x = 5$ $y = 1 + \pi$.

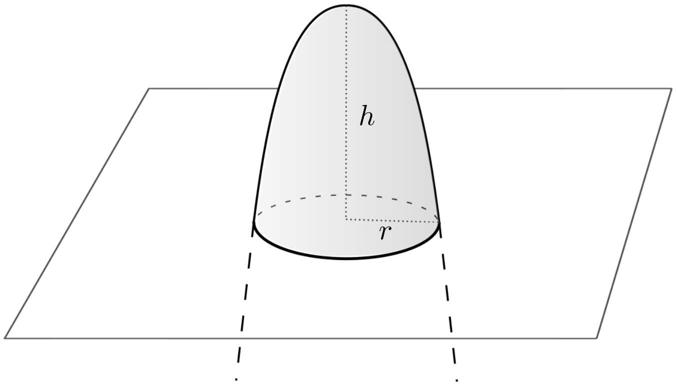
f. [2 points] At which value(s) of $-6 < x < 6$ does the function $f(x)$ have an inflection point?

Solution: The only point where the function $f(x)$ changes concavity is at $x = 5$. **Answer:** $x = 5$

7. [11 points] A group of meteorologists observe that the sea level is rising by observing a piece of a rock in the sea.

Only the tip of the rock is visible, and as the sea water rises, less and less of the rock is above water. Let h and r be the height and radius (in inches), respectively, of the part of the rock that is above the sea. The volume of the rock (in cubic inches) is then given by the formula

$$V = \frac{\pi}{2}(1+r^2)h.$$



a. [8 points] The meteorologists notice that, as the level of the sea is rising, the radius and volume of the rock are changing. A year after they started taking the measurements, the radius and height of the rock are 5 and 46 inches, respectively. They notice that at that time, the radius is decreasing at a rate of 0.05 inches per year, which makes the volume change at a rate of 80 cubic inches per year. At what rate is the height of the rock changing at that time? Be sure to include units.

Solution:

$$V = \frac{\pi}{2}(1+r^2)h$$
$$\frac{dV}{dt} = \frac{\pi}{2}\left(2r\frac{dr}{dt}\right)h + \frac{\pi}{2}(1+r^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \pi r h\frac{dr}{dt} + \frac{\pi}{2}(1+r^2)\frac{dh}{dt}$$

$$-80 = \pi(5)(46)(-0.05) + \frac{\pi}{2}(1+(5)^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-80+11.5\pi}{13\pi} \approx -1.07$$

Answer: The height of the part of the rock that is above the sea is (circle one)

Increasing [✓ Decreasing] Not Enough Information

at a rate of 1.07 inches per year.

b. [3 points] Meteorologists discover that the sea level increases as a function of the average temperature registered during the year. Let $R(T)$ be the sea level (in centimeters) if the average temperature next year is T degrees Celsius (°C). Circle the one statement below that is best supported by the equation

$$R'(16) = 0.5.$$

(I) The sea level will rise by about 0.5 centimeters if the average temperature next year is 16 degrees Celsius.

(II) [✓ The sea level will rise by approximately 0.1 centimeters if the average temperature next year rises from 15.8°C to 16°C next year.]

(III) The average temperature next year needs to increases by about 0.5°C in order for the sea level to rise to a level of 16 centimeters next year.

(IV) The sea level will rise by about 0.2 centimeters if the average temperature next year rises from 16°C to 16.2°C next year.

(V) The sea level rises by 0.5 centimeters for every additional degree Celsius the average temperature rises above 16°C.

8. [6 points] A botanist is studying the number of offspring plants that will survive for a year in her experimental crop. She notices that the number of offspring depends on the number of seeds sown in the crop. Let $P(s)$ be the number of plants that will survive after a year in the crop if s thousand seeds are sown in the crop. The formula for $P'(s)$ is

$$P'(s) = \frac{1000(1-ks)(1+ks)}{(1+(ks)^2)^2} \qquad \text{for} \quad 0 \le s < \infty,$$

where $k \ge \frac{1}{2}$ is a constant.

a. [6 points] Find and classify all the critical points of $P(s)$ on $(0, \infty)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write "NONE" if appropriate. Your answer may depend on the constant k.

Solution: Critical points only when $P'(s) = 0$. That is when $1000(1-ks)(1+ks) = 0$. Solving for s we get $s = -\frac{1}{k}$ and $s = \frac{1}{k}$. Only $s = \frac{1}{k}$ is positive.

Classification of $s = \frac{1}{k}$:

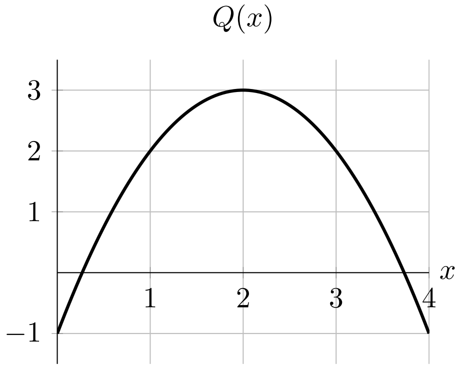
- For $0 \le s < \frac{1}{k}$ we can pick $s = 0$ and obtain $P'(0) = 1000 > 0$.
- For $\frac{1}{k} < s$ we can pick $s = \frac{2}{k}$ and obtain $P'\left(\frac{2}{k}\right) = \frac{1000(1-2)(1+2)}{(1+(2)^2)^2} = -120 < 0$.

Therefore, $s = \frac{1}{k}$ is a local maximum.

Answer: Critical point(s) at $s = \dfrac{1}{k}$

Local maximum(s) at $s = \dfrac{1}{k}$ Local minimum(s) at $s =$ NONE

9. [11 points] Let $Q(x) = -(x-2)^2 + 3$ be the quadratic approximation of the function $y = f(x)$ at $x = 3$. A part of the graph of $Q(x)$ is shown below.



a. [6 points] If possible, find the following quantities exactly. If there is not enough information to obtain an **exact** answer, write "NEI".

Solution:

$$f''(3) = \ -2, \quad f'''(3) = \ \text{NEI}, \quad f(0) = \ \text{NEI},$$

$$Q''(3) = \ -2, \qquad Q'''(3) = \ 0, \qquad Q(0) = \ -1.$$

b. [4 points] Assume that the function $f(x)$ is invertible and let $g(y) = f^{-1}(y)$ be its inverse. Given that $f(3) = 2$, find the linear approximation $L(y)$ of $g(y)$ at $y = 2$. Your answer should not include the letters f or g. Show all your work.

Solution: The formula for $L(y)$ is given by

$$L(y) = f^{-1}(2) + (f^{-1})'(2)(y-2).$$

We know that $f^{-1}(2) = 3$ and $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. Hence

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(3)} = -\frac{1}{2}$$

Answer: $L(y) = 3 - \frac{1}{2}(y-2)$

c. [1 point] Use the linear approximation $L(y)$ to approximate a solution to the equation $f(x) = 1.7$.

Solution:

Answer: $L(1.7) = 3 - \frac{1}{2}(1.7-2) = 3.15.$