Math 115 — First Midterm — February 12, 2019

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	5	
3	8	
4	10	
5	15	

Problem	Points	Score
6	12	
7	12	
8	8	
9	8	
10	9	
Total	100	

1. [13 points] Alex and Misha are running a 100 meter race at Ferry Field. Alex starts the race at the starting line, but because he's on the track team, he lets Misha start the race several meters in front of the starting line.

Five seconds after the race begins, Misha is 32 meters from the starting line. Sixteen seconds into the race, Misha is 92 meters from the starting line, but Alex is crossing the finish line and wins the race.

a. [2 points] Write a formula for $A(t)$, Alex's distance from the starting line, in meters, assuming that $A(t)$ is a *linear* function.

Solution: We are given that $A(0) = 0$ and $A(16) = 100$, so the vertical intercept is 0 and the slope is $\frac{100-0}{16-0} = \frac{100}{16} = 6.25$.

Answer: $A(t) =$ $\frac{100}{16}t$

b. [3 points] Write a formula for $M(t)$, Misha's distance from the starting line, in meters, assuming that $M(t)$ is an *exponential* function.

Solution: We are given that $M(5) = 32$ and $M(16) = 92$, and that $M(t) = ab^t$. Then

$$92 = ab^{16}$$
$$32 = ab^5$$
$$\frac{92}{32} = b^{11}$$
$$b = \left(\frac{92}{32}\right)^{\frac{1}{11}} \approx 1.10 \quad \text{and} \quad a = \frac{32}{b^5} = \frac{32}{\left(\frac{92}{32}\right)^{\frac{5}{11}}} \approx 19.8.$$

Answer: $M(t) =$ $19.8(1.10)^t$

c. [2 points] How many meters in front of the starting line did Misha start the race?

Answer: 19.8

d. After their race, Alex and Misha walk north to get some lunch in town. Part way through this trip, Alex realizes he left his watch at the field, and so they turn around and walk south until they return to Ferry Field. They get Alex's watch and then walk north again. Their distance in miles from Ferry Field, t minutes after they leave for the first time, is given by the differentiable function $D(t)$. Some values for $D(t)$ are provided in the table below.

t	3	6	10	12	13	15	19	24
$D(t)$	0.15	0.32	0.57	0.40	0.27	0.03	0.21	0.43

i. [2 points] What is their average velocity between $t = 10$ and $t = 19$? Include units.

Solution: average velocity $= \dfrac{0.21 - 0.57}{19 - 10} = -0.04$ miles per min

ii. [2 points] Estimate their instantaneous velocity 16 minutes into their trip. Include units.

Solution: instantaneous velocity at $t = 16 \approx \dfrac{0.21 - 0.03}{19 - 15} = 0.045$ miles per min

iii. [2 points] On which of the following interval(s) **must** $D'(t)$ be less than or equal to zero for **all** values of t in the interval? Circle all correct choices.

[3, 6] [6, 10] [10, 12] $\boxed{[12, 13]}$ [13, 15] NONE OF THESE

2. [5 points] The logistic function, which is frequently used in machine learning applications, is given by the formula

$$S(r) = \frac{1}{1+e^{-2r}}.$$

Use the limit definition of the derivative to write an explicit expression for $S'(3)$. *Your answer should not involve the letter S. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Solution:

$$S'(3) = \lim_{h\to 0} \frac{\dfrac{1}{1+e^{-2(3+h)}} - \dfrac{1}{1+e^{-6}}}{h}.$$

3. [8 points] A U of M student is studying the various invasive species of insects she finds in a sample plot of forest.

a. [3 points] On her first visit to the plot, she finds 18 ash borers. She estimates that the number of ash borers will grow by 2.4% each day. Based on this estimate, write a formula for $A(w)$, the number of ash borers in her plot w weeks after her first visit.

Solution: We know that $A(w)$ is of the form ab^w and that $A(0) = a = 18$. We also know that after 1 day, which is $\frac{1}{7}$ of a week, the number of ash borers is 18(1.024). That is,

$$A(1/7) = 18b^{1/7} = 18(1.024)$$

$$b^{1/7} = 1.024$$

$$b = 1.024^7$$

Answer: $A(w) =$ $18(1.024)^{7w}$

b. [5 points] The student's data suggest that, w weeks after her first visit, the number of pineshoot beetles in her plot will be given by

$$P(w) = 11e^{w/6},$$

while the number of gypsy moths will be given by

$$G(w) = 3(1.37)^w.$$

After her first visit, how many weeks will it take for the number of pineshoot beetles to equal the number of gypsy moths? Give your answer in **exact form**.

Solution:

Method 1:

$$11e^{w/6} = 3(1.37)^w$$

$$\ln\left(11e^{w/6}\right) = \ln\left(3(1.37)^w\right)$$

$$\ln(11) + \ln\left(e^{w/6}\right) = \ln(3) + \ln\left(1.37^w\right)$$

$$\ln(11) + w/6 = \ln(3) + w\ln(1.37)$$

$$\ln(11) - \ln(3) = w(\ln(1.37) - 1/6)$$

$$w = \frac{\ln(11) - \ln(3)}{\ln(1.37) - 1/6}$$

Method 2:

$$11e^{w/6} = 2(1.37)^w$$

$$\frac{11}{3} = \frac{1.37^w}{e^{w/6}}$$

$$\frac{11}{3} = \left(\frac{1.37}{e^{1/6}}\right)^w$$

$$\ln\left(\frac{11}{3}\right) = \ln\left(\left(\frac{1.37}{e^{1/6}}\right)^w\right)$$

$$\ln\left(\frac{11}{3}\right) = w\ln\left(\frac{1.37}{e^{1/6}}\right)$$

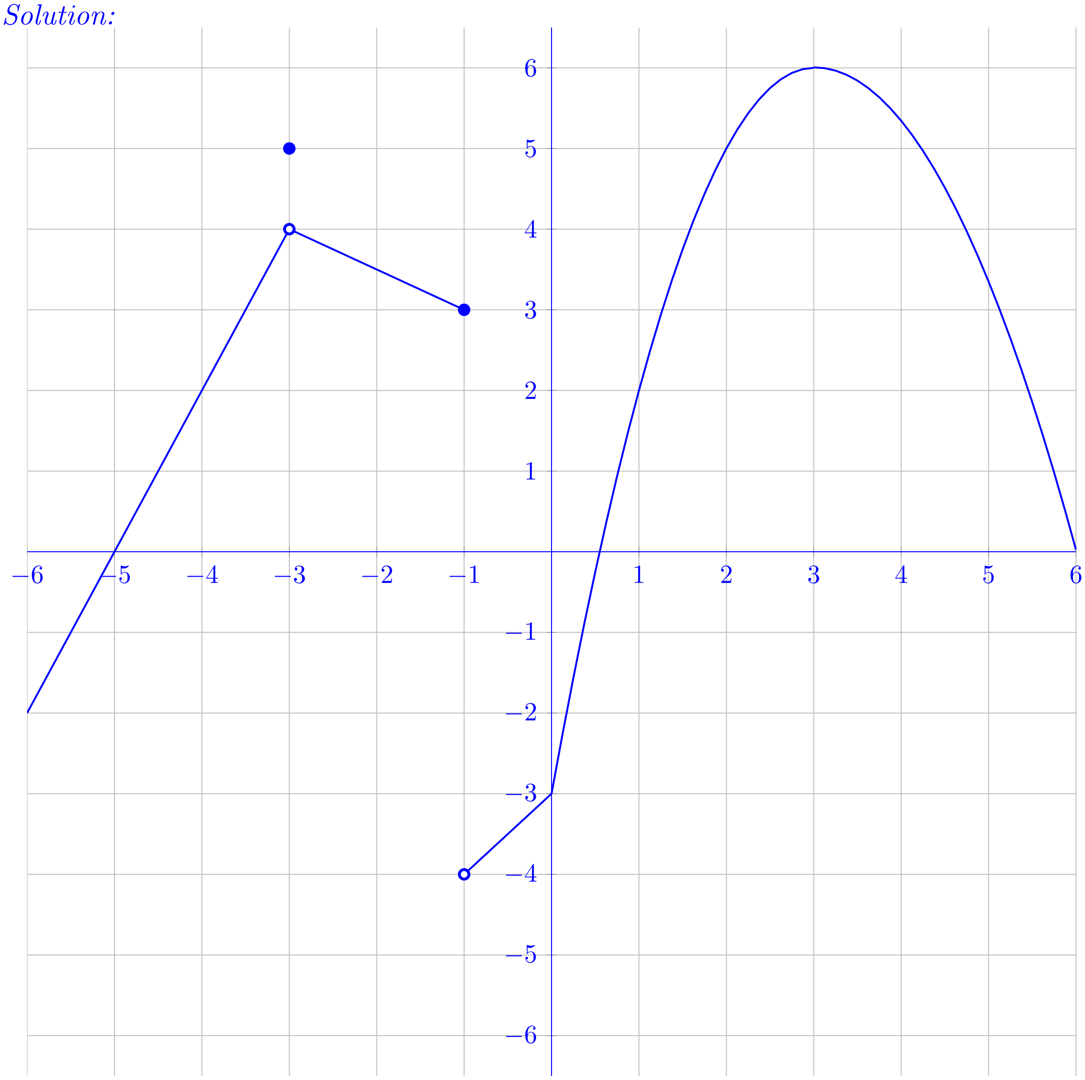
$$w = \frac{\ln\left(\frac{11}{3}\right)}{\ln\left(\frac{1.37}{e^{1/6}}\right)}$$

Answer: $w = \dfrac{\ln(11) - \ln(3)}{\ln(1.37) - 1/6}$

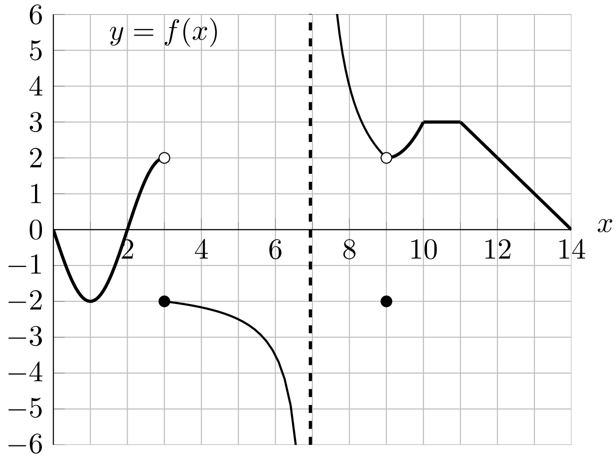
4. [10 points] On the axes provided below, sketch the graph of a single function $f(x)$ that satisfies all of the following conditions:

- the function $f(x)$ has domain $-6 \le x \le 6$
- $f(0) = -3$
- $f(x)$ is continuous everywhere except at $x = -3$ and $x = -1$
- $f'(x) = 2$ for $-6 < x < -4$
- $\lim_{x \to -3} f(x) = 4$
- $\lim_{x \to -1^-} f(x) = 3$
- $\lim_{x \to -1^+} f(x) = -4$
- the average rate of change of $f(x)$ from $x = 0$ to $x = 6$ is $\frac{1}{2}$
- $f'(3) = 0$
- $f(x)$ is decreasing from $x = 4$ to $x = 6$

Solution:



5. [15 points]
Below is a portion of the graph of an **odd** function $f(x)$, and the formula for a function $g(x)$. Note that $f(x)$ is linear for $11 < x < 14$.



$$g(x) = \frac{x^4 + 1}{e^{x^2}}$$

In the following parts, evaluate each of the given quantities. If the value does not represent a real number (including the case of limits that diverge to ∞ or $-\infty$), write DNE or "does not exist." You do not need to show work in this problem. Give your answers in **exact form**.

a. [2 points] $g(f(2))$

Answer: = 1

b. [2 points] $\lim_{x \to 7} f(x)$

Answer: = DNE

c. [2 points] $\lim_{x \to -1} \Big(f(x) + g(x) \Big)$

Answer: = $2 + \frac{2}{e}$

d. [2 points] $\lim_{x \to \infty} g(x)$

Answer: = 0

e. [2 points] $\lim_{h \to 0} \frac{f(12 + h) - 2}{h}$

Answer: = -1

f. [2 points] $\lim_{x \to -9} f(x)$

Answer: = -2

g. [2 points] $\lim_{x \to 11^+} f(f(x))$

Answer: = 2

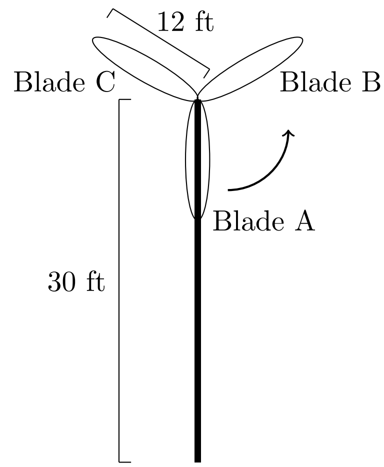
h. [1 point] $\lim_{x \to 3} g(f(x))$

Answer: = $\frac{17}{e^4}$

6. [12 points]

a. A wind turbine, spinning <u>counterclockwise</u> at a constant rate, stands 30 feet tall from the ground to its central spinning axis. It has three equally spaced blades, each 12 feet long (as measured from the center of the axis). These blades are labeled as Blade A, B, and C in the figure. At exactly 1:00 pm, an engineer sees that Blade A is pointing straight toward the ground as shown. Blade A then takes exactly 1.5 seconds to return to this downward position. Let $A(t)$ be the height from the ground, in feet, of the outermost tip of Blade A, t seconds after 1:00 pm.

i. [4 points] Write a formula for the trigonometric function $A(t)$.



Solution:

$$A(t) = -12\cos\left(\frac{4\pi}{3}t\right) + 30$$

ii. [3 points] The height $C(t)$ of the outermost tip of Blade C, in feet above the ground, can be given as a transformation of $A(t)$. Circle **all** correct transformations below.

[C(t) = A(t - 0.5)]	$C(t) = A(t - 2\pi/3)$	$C(t) = A(t) + 18$	$C(t) = A(t - 1)$
$C(t) = A(t + 0.5)$	$C(t) = A(t + 2\pi/3)$	$C(t) = A(t) - 18$	[C(t) = A(t + 1)]

b. [5 points] The height in feet of the tip of one of the blades on a *different* windmill, t seconds after 1:00pm, is given by

$$W(t) = 24\cos\left(\frac{\pi}{3}t\right) + 60.$$

Find the first two positive times, in seconds, where $W(t) = 40$. Give your answers in **exact form**.

Solution:

$$24\cos\left(\frac{\pi}{3}t\right) + 60 = 40$$

$$\cos\left(\frac{\pi}{3}t\right) = \frac{-20}{24}$$

$$\frac{\pi}{3}t_1 = \cos^{-1}\left(\frac{-20}{24}\right)$$

$$t_1 = \frac{3}{\pi}\cos^{-1}\left(\frac{-20}{24}\right).$$

Now, we use the symmetry of the cosine function. Since the period is 6 and there is no horizontal shift,

$$t_2 = 6 - \frac{3}{\pi}\cos^{-1}\left(\frac{-20}{24}\right)$$

Answer: $t =$ $\frac{3}{\pi}\cos^{-1}(\frac{-20}{24})$ and $t =$ $6 - \frac{3}{\pi}\cos^{-1}(\frac{-20}{24})$

7. [12 points]
The annual number of respiratory infections in a city is a function of the amount of carbon in the atmosphere above that city.

Let $R(p)$ be the annual number of respiratory infections in Ann Arbor when there are p thousand tons of carbon in the atmosphere above the city.

Let $C(k)$ be the healthcare cost, in thousands of dollars, of treating k respiratory infections.

The functions $R(p)$ and $C(k)$ are differentiable and invertible.

a. [3 points] Give a practical interpretation of the equation $R^{-1}(212) = 24$.

Solution: There are 212 respiratory infections annually in Ann Arbor when there are 24 thousand tons of carbon in the atmosphere above the city.

b. [3 points] Give a practical interpretation of the equation $C(R(17)) = 650$.

Solution: When there are 17 thousand tons of carbon in the atmosphere above Ann Arbor, the resulting annual healthcare cost for respiratory infections in Ann Arbor is 650 thousand dollars.

c. [3 points] Write a mathematical equation that represents the following statement:

The healthcare cost of treating 165 respiratory infections is 400 thousand dollars more than the healthcare cost of treating 130 respiratory infections.

Answer: $C(165) = 400 + C(130)$

d. [3 points] Complete the following sentence using the fact that $R'(38) = 4$:

If the amount of carbon in the atmosphere above Ann Arbor is reduced from 41 *thousand tons to* 38 *thousand tons, ...*

Solution: then the annual number of respiratory infections in Ann Arbor will decrease by approximately 12.

8. [8 points]

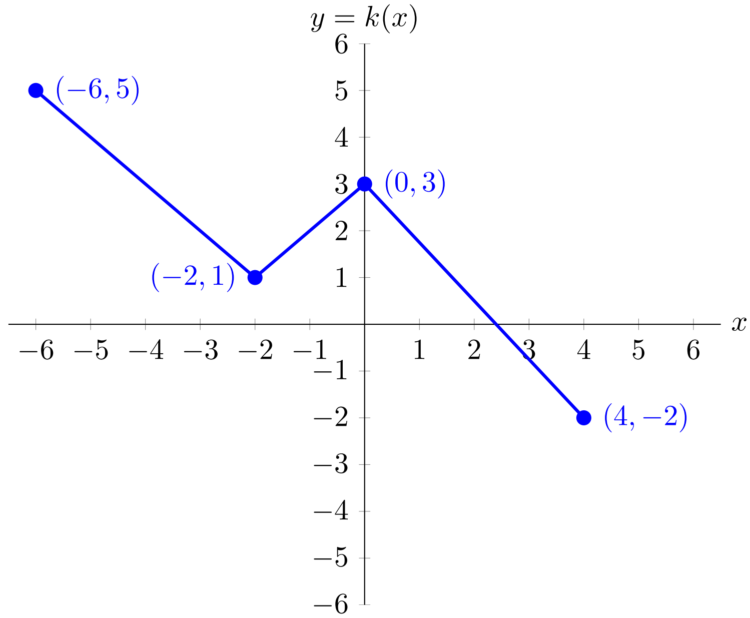
a. [5 points] The function $h(x)$, with domain $-3 \le x \le 2$, has the table of values shown below. Also, $h(x)$ is linear between each consecutive pair of points in the table.

x	-3	-1	0	2
$h(x)$	-4	0	-2	3

Consider the function

$$k(x) = -h\left(\frac{1}{2}x\right) + 1.$$

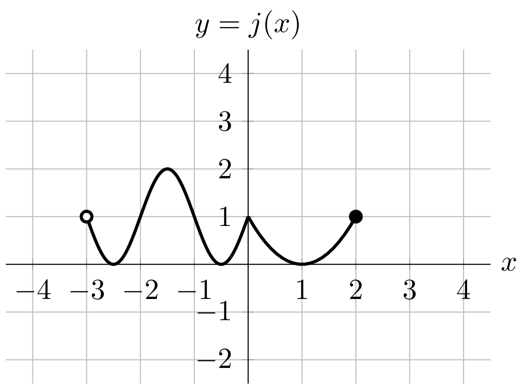
Find the domain and range of $k(x)$, and then carefully sketch the entire graph of $k(x)$ on the given axes. Make your graph large and unambiguous, and be sure that the coordinates of important points are clear.

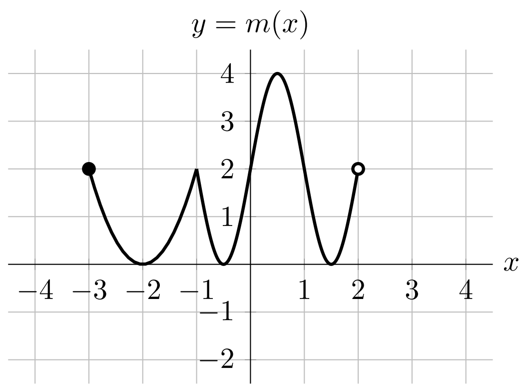


Domain: $\underline{-6} \le x \le \underline{4}$

Range: $\underline{-2} \le y \le \underline{5}$

b. [3 points] Below is the graph of a function $j(x)$. Also shown is the graph of $m(x)$, which was obtained from $j(x)$ through one or more transformations. Find a formula for $m(x)$ in terms of the function $j(x)$.





Answer: $m(x) = \underline{2j(-(x+1))}$

9. [8 points] The function $r(x)$ is given by the following formula, where c is a positive constant:

$$r(x) = \begin{cases} \dfrac{3x+3}{(x+5)(x-2)} & x < 0 \\ \dfrac{c}{x^3-1} & 0 \le x < 4 \\ \sqrt{2-\dfrac{8}{x}} & 4 \le x. \end{cases}$$

It is not necessary to show work in this problem.

a. [2 points] Find $\lim_{x\to-\infty} r(x)$. If the limit does not exist (including the case of limits that diverge to ∞ or $-\infty$), write DNE.

Answer: $\lim_{x\to-\infty} r(x) =$ 0

b. [2 points] For what value(s) of x does $r(x)$ have a vertical asymptote? Write NONE if there are no such values.

Answer(s): $x =$ $-5, 1$

c. [2 points] For what value(s) of x is $r(x) = 0$? Write NONE if there are no such values.

Answer(s): $x =$ $-1, 4$

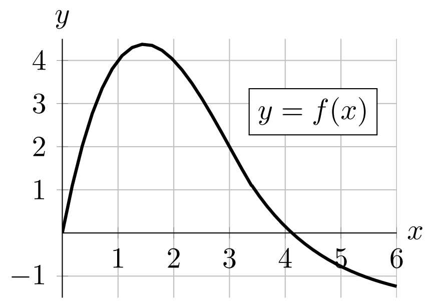
d. [2 points] For what value(s) of c is the function $r(x)$ continuous at $x = 0$? Write NONE if there are no such values.

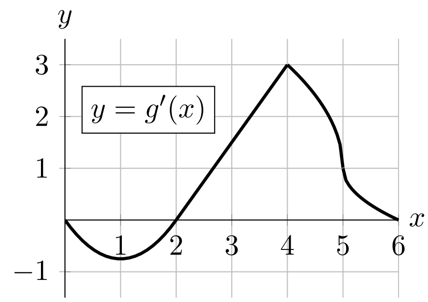
Solution: We plug in zero to the first two pieces of $r(x)$ and set those expressions equal to each other:

$$\frac{3}{(5)(-2)} = \frac{c}{-1}$$
$$c = \frac{3}{10}$$

Answer(s): $c =$ $\frac{3}{10}$

10. [9 points] The graph of the function $f(x)$ and the graph of $g'(x)$ (the derivative of the function g) are given below.





a. [3 points] Rank these five quantities in order from least to greatest by filling in the blanks below with the options I–V.

I. the average rate of change of $f(x)$ from $x = 1$ to $x = 3$

II. $f'(3)$

III. the slope of the tangent line to $f(x)$ at $x = 1$

IV. $\frac{f(5) - f(0)}{5}$

V. the number 0

II < **I** < **IV** < **V** < **III**

b. [2 points] On which of the following interval(s) is $f(x)$ invertible? Circle all correct choices.

(0,2) (1,4) [$(2, 6)$] NONE OF THESE

c. [2 points] On which of the following interval(s) is $g(x)$ increasing? Circle all correct choices.

(1,3) [$(2, 5)$] [$(5, 6)$] NONE OF THESE

d. [2 points] At which value of x, for $0 < x < 6$, is the slope of $g(x)$ most negative?

Answer: $x =$ 1