Math 115 — Second Midterm — March 26, 2019

EXAM SOLUTIONS

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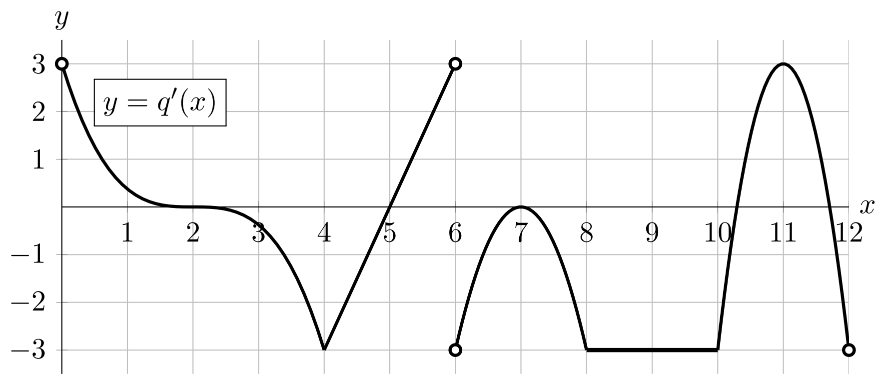
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 10 problems.
Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	16	
3	11	
4	10	
5	8	

Problem	Points	Score
6	10	
7	8	
8	8	
9	9	
10	8	
Total	100	

1. [12 points] The function $q(x)$ is continuous on $[0, 12]$. The graph of $q'(x)$ (the derivative of q) is given below.



a. [2 points] On which of the following interval(s) is $q(x)$ decreasing? Circle all correct choices.

(0,2) $\boxed{(6,7)}$ $\boxed{(7,8)}$ NONE OF THESE

b. [2 points] On which of the following interval(s) is $q(x)$ concave down? Circle all correct choices.

$\boxed{(0,2)}$ $\boxed{(2,4)}$ (6,7) NONE OF THESE

c. [2 points] Which of the following are critical point(s) of $q'(x)$? Circle all correct choices.

$\boxed{x = 2}$ $x = 5$ $\boxed{x = 9}$ NONE OF THESE

d. [2 points] Which of the following are critical point(s) of $q(x)$? Circle all correct choices.

$\boxed{x = 5}$ $\boxed{x = 6}$ $x = 11$ NONE OF THESE

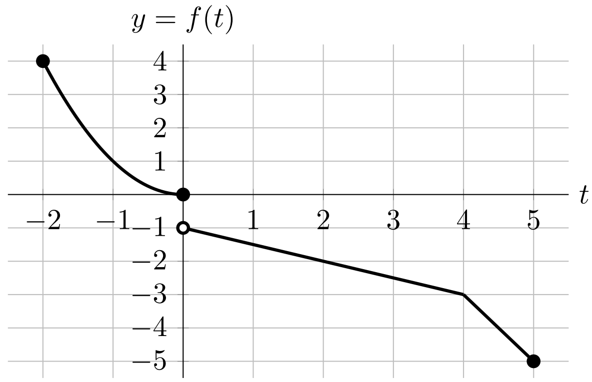
e. [2 points] At which of the following value(s) of x does $q(x)$ have a local maximum? Circle all correct choices.

$\boxed{x = 6}$ $x = 7$ $x = 11$ NONE OF THESE

f. [2 points] At which of the following value(s) of x does $q(x)$ have an inflection point? Circle all correct choices.

$x = 2$ $\boxed{x = 4}$ $\boxed{x = 7}$ NONE OF THESE

2. [16 points]
Shown to the right is the graph of a function $f(t)$.



Note that:

- $f(t) = t^2$ on $[-2, 0]$,
- $f(t)$ is linear on the intervals (0,4) and (4,5).

a. Evaluate each of the following quantities **exactly**, or write DNE if the value does not exist. You do not need to show work, but limited partial credit may be awarded for work shown.

i. [2 points] Find $(f^{-1})'(-2)$.

Solution: $(f^{-1})'(-2) = \dfrac{1}{f'(f^{-1}(-2))} = \dfrac{1}{f'(2)} = \dfrac{1}{-1/2} = -2$

Answer: $(f^{-1})'(-2) =$ -2

ii. [2 points] Let $g(t) = \sin(t)f(t)$. Find $g'(4)$.

Solution: We have $g'(t) = \sin(t)f'(t) + \cos(t)f(t)$, so $g'(4) = \sin(4)f'(4) + \cos(4)f(4)$. Since $f'(4)$ DNE, $g'(4)$ DNE. (This is because, if $g'(4)$ existed, then since $f(t) = g(t)/\sin(t)$, we would have that $f'(4) = (\sin(4)g'(4) - g(4)\cos(4))/\sin^2(4)$ also existed.)

Answer: $g'(4) =$ **DNE**

iii. [4 points] Let $h(t) = \dfrac{f(2t+2)}{2^t}$. Find $h'(0)$.

Solution: We have $h'(t) = \dfrac{2^t f'(2t+2)\cdot 2 - f(2t+2)\ln(2)2^t}{(2^t)^2}$, so
$h'(0) = \dfrac{2^0 f'(2)\cdot 2 - f(2)\ln(2)2^0}{(2^0)^2} = 1(-1/2)(2) - (-2)\ln(2) = -1 + 2\ln(2)$.

Answer: $h'(0) =$ $-1 + 2\ln(2)$

iv. [4 points] Let $j(t) = \ln(-f'(t))$. Find $j'(-1)$.

Solution: We have $j'(t) = \dfrac{1}{-f'(t)}(-f''(t))$.
Since $[t^2]' = 2t$ and $[t^2]'' = 2$, we have $f'(-1) = -2$ and $f''(-1) = 2$, so
$j'(-1) = \dfrac{1}{-f'(-1)}(-f''(-1)) = \dfrac{1}{-(-2)}(-2) = -1$.

Answer: $j'(-1) =$ **-1**

b. [2 points] On which of the following interval(s) does $f(t)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct choices.

$[-2, 5]$ $\quad$ [0,3] $\quad$ [3,5] $\quad$ $\boxed{\text{NONE OF THESE}}$

c. [2 points] On which of the following interval(s) does $f(t)$ satisfy the conclusion of the Mean Value Theorem? Circle all correct choices.

$\boxed{[-2, 5]}$ $\quad$ $[0, 3]$ $\quad$ $[3, 5]$ $\quad$ NONE OF THESE

3. [11 points] Suppose $h(x)$ is a continuous function defined for all real numbers x. The derivative and second derivative of $h(x)$ are given by

$$h'(x) = \frac{2x}{3(x^2-1)^{2/3}} \qquad \text{and} \qquad h''(x) = -\frac{2(x^2+3)}{9(x^2-1)^{5/3}}.$$

a. [6 points] Find the x-coordinates of all critical points of $h(x)$ and all values of x at which $h(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: The critical points occur at values of x where $h'(x) = 0$ or $h'(x)$ does not exist. We have $h'(x) = 0$ at $x = 0$, and $h'(x)$ does not exist at -1 and 1. Since h is continuous for all real numbers, these three values are all in its domain and so all are critical points.

Now we need to know the sign of $h'(x)$ on four intervals. To determine this, we consider the signs of the numerator and denominator.

	sign of $2x$	sign of $3(x^2-1)^{2/3}$	sign of $h'(x)$
$-\infty < x < -1$	$-$	$+$	$-$
$-1 < x < 0$	$-$	$+$	$-$
$0 < x < 1$	$+$	$+$	$+$
$1 < x < \infty$	$+$	$+$	$+$

Thus $h'(x)$ is decreasing for $x < 0$ and increasing for $x > 0$, so we have a local minimum at $x = 0$ and no local maxima.

Answer: Critical point(s) at $x =$ **-1,0,1**

Answer: Local max(es) at $x =$ **NONE**

Answer: Local min(s) at $x =$ **0**

b. [5 points] Find the x-coordinates of all inflection points of $h(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: Inflection points can only occur at values of x where $h''(x) = 0$ or $h''(x)$ does not exist. There are no places where $h''(x) =$), and $h''(x)$ does not exist at -1 and 1. Since h is continuous for all real numbers, these three values are all candidates for inflection points.

Now we need to know the sign of $h''(x)$ on three intervals. To determine this, we consider the signs of the numerator and denominator, plus we take into account the negative sign in front.

	$-$ sign in front	sign of $2(x^2+3)$	sign of $9(x^2-1)^{5/3}$	sign of $h'(x)$
$-\infty < x < -1$	$-$	$+$	$+$	$-$
$-1 < x < 1$	$-$	$+$	$-$	$+$
$1 < x < \infty$	$-$	$+$	$+$	$-$

Thus $h''(x)$ changes sign at -1 and 1, so $h(x)$ changes concavity at both of these points.

Answer: Inflection point(s) at $x =$ **-1,1**

4. [10 points]

a. Let $\mathcal{C}$ be the curve given by the equation

$$y\cos(2x) = y^3 + b,$$

where b is a constant. The curve $\mathcal{C}$ passes through the point $(0, 2)$.

i. [2 points] Find b.

Solution: Plugging in $(0, 2)$, we find that

$$2\cos(2\cdot 0) = 2^3 + b$$

$$2 = 8 + b$$

$$b = -6.$$

Answer: $b =$ **-6**

ii. [5 points] For the curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution:
Using implicit differentiation, and the product rule on the left-hand side,

$$\begin{aligned} -y\sin(2x)\cdot 2 + \frac{dy}{dx}\cos(2x) &= 3y^2\frac{dy}{dx} \\ \frac{dy}{dx}\cos(2x) - 3y^2\frac{dy}{dx} &= 2y\sin(2x) \\ \frac{dy}{dx}\left(\cos(2x) - 3y^2\right) &= 2y\sin(2x) \end{aligned}$$

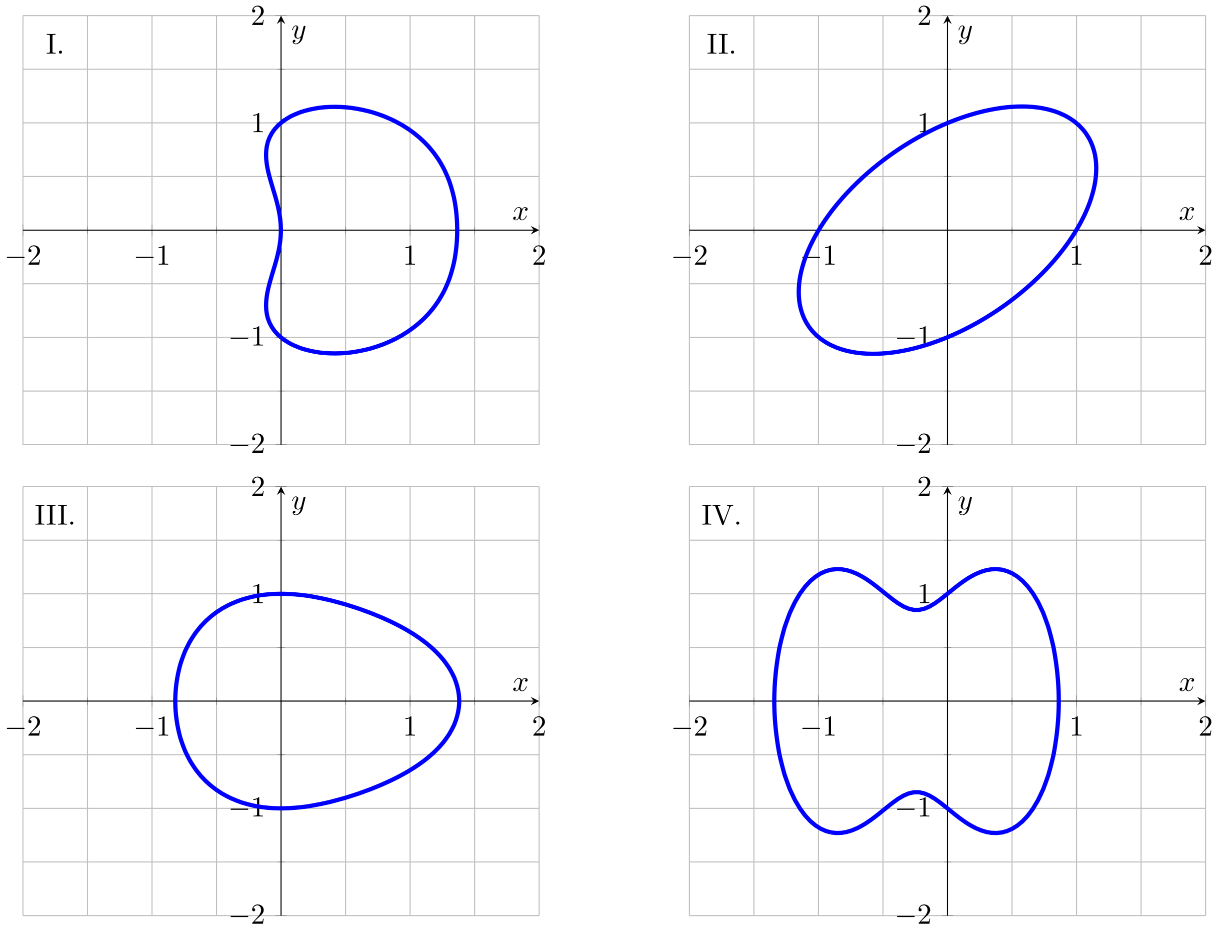
$$\frac{dy}{dx} = \frac{2y\sin(2x)}{\cos(2x) - 3y^2}$$

Answer: $\frac{dy}{dx} =$ $\frac{dy}{dx} = \frac{2y\sin(2x)}{\cos(2x) - 3y^2}$

b. [3 points] A different curve $\mathcal{R}$ passes through the point $(0, 1)$ and satisfies

$$\frac{dy}{dx} = \frac{2x - y}{x - 2y}.$$

One of the following graphs is the graph of $\mathcal{R}$. Which of the graphs is it? Write the numeral (I, II, III, or IV) of the graph you choose on the answer line at the bottom of this page.



Solution: We find that the slope at the given point $(0, 1)$ is $1/2$, so this rules out III. Finding that the slope at the point $(0, -1)$ must also be $1/2$, we conclude that II must be correct. (We could also have ruled out I and IV (and III) by noting that these graphs have vertical tangents when $y = 0$, but dy/dx is not undefined when $y = 0$.)

Answer: **II**

5. [8 points] Consider the function $h(x)$ where k and A are constants:

$$h(x) = \begin{cases} 2x+1 & x \le k \\ (x-A)^2+2 & x > k \end{cases}$$

a. [5 points] There is exactly one choice of the constants A and k that make $h(x)$ differentiable. Find these values of A and k.

Solution: Since both pieces of the function are differentiable, the only place where $h(x)$ might not be differentiable is at the point $x = k$. To be differentiable at $x = k$, the function must first be continuous at $x = k$, so we set the two parts of the piecewise function equal to each other at $x = k$:

$$2k + 1 = (k - A)^2 + 2.$$

We also need to set their derivatives equal to each other at $x = k$:

$$\begin{aligned} 2 &= 2(k-A) \\ 2 &= 2k - 2A \\ 2 + 2A &= 2k \\ 1 + A &= k. \end{aligned}$$

Substituting this into the first equation we get

$$\begin{aligned} 2(1+A)+1 &= ((1+A)-A)^2+2 \\ 2+2A+1 &= (1)^2+2 \\ 2A+3 &= 3 \\ 2A &= 0 \\ A &= 0, \end{aligned}$$

and since $k = 1 + A$ we have that $k = 1$.

Answer: $A =$ 0 **Answer:** $k =$ 1

b. [3 points] If $A > k$, then $h(x)$ has two critical points. What are the x-coordinates of these points? Your answers may be in terms of A and/or k. Show work or briefly explain your reasoning.

Solution: If $A > k$, then we certainly cannot be in the situation where $A = 0$ and $k = 1$ from part a. Thus $h(x)$ will not be differentiable at k and $x = k$ is one of the critical points.

Since the derivative of $2x + 1$ is 2, there are no critical points less than k.

Finally, the derivative of $(x - A)^2 + 2$ is $2(x - A)$, which is zero when $x = A$. Since $A > k$, this point does fall in the domain of the second piece, and so $x = A$ is the second critical point.

Answer: Critical point(s) at $x =$ k **and** A

6. [10 points]

a. [4 points] Below is a table of values for a differentiable function $g(x)$. Also shown are some values of $g'(x)$, which is an increasing function and also differentiable.

x	3	8	10
$g(x)$	10	1	0
$g'(x)$	-4	0.6	2

i. [2 points] Write a formula for $L(x)$, the linear approximation of $g(x)$ at $x = 3$.

Solution: The general formula for the linear approximation is $L(x) = g(a) + g'(a)(x - a)$. In our case, $a = 3$, $g(3) = 10$, and $g'(3) = -4$. So $L(x) = 10 - 4(x - 3)$.

Answer: $L(x) =$ $10 - 4(x - 3)$

ii. [1 point] Use your formula for $L(x)$ to estimate $g(3.2)$.

Solution: $g(3.2) \approx L(3.2) = 10 - 4(3.2 - 3) = 10 - 4(.2) = 9.2$

Answer: $g(3.2) \approx$ 9.2

iii. [1 point] Is your estimate of $g(3.2)$ an overestimate or an underestimate? Circle your answer.

Solution: Since $g'(x)$ is an increasing function, $g''(x)$ is positive, and $g(x)$ is concave up. The linear approximation of a concave up function is always an underestimate.

Overestimate $\boxed{\textit{Underestimate}}$ Cannot be determined

b. [2 points] The quadratic approximation of $g(x)$ at $x = 10$ is

$$Q(x) = 2(x - 10) + 2(x - 10)^2.$$

Find $g''(10)$.

Solution: Since we know $Q(x) = g(10) + g'(10)(x - 10) + \dfrac{g''(10)}{2}(x - 10)^2$, we must have $\dfrac{g''(10)}{2} = 2$, and $g''(10) = 4$.

Answer: $g''(10) =$ **4**

c. [4 points] Let $h(x) = (g(x))^3$. The linear approximation of $h(x)$ at $x = 6$ is

$$K(x) = 8 + 3(x - 6).$$

Find $g(6)$ and $g'(6)$.

Solution: We know $K(x) = h(6) + h'(6)(x - 6)$, so $h(6) = (g(6))^3 = 8$, and $g(6) = 2$. Now we find $g'(6)$. By the chain rule we have $h'(x) = 3(g(x))^2 g'(x)$. Since $h'(6) = 3$, we have $3(g(6))^2 g'(6) = 3$, or $3(2)^2 g'(6) = 3$. That is, $g'(6) = 1/4$.

Answer: $g(6) =$ **2** **Answer:** $g'(6) =$ **1/4**

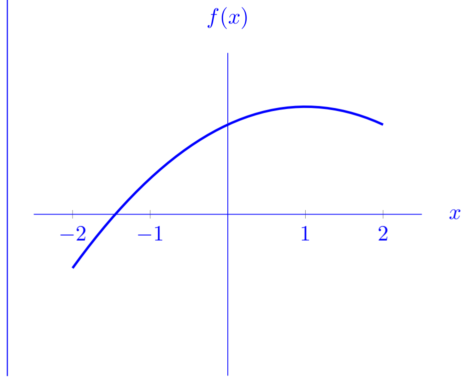
7. [8 points] For each part, draw a function on the given axes that satisfies the given conditions. Or, if no such function exists, write DNE and provide a brief explanation.

Make sure your graphs are unambiguous and that the domain of each graph is clear.

a. [2 points]

A differentiable function $f(x)$ with domain $[-2, 2]$ that has a global maximum at $x = 1$ and $f''(x) \leq 0$.

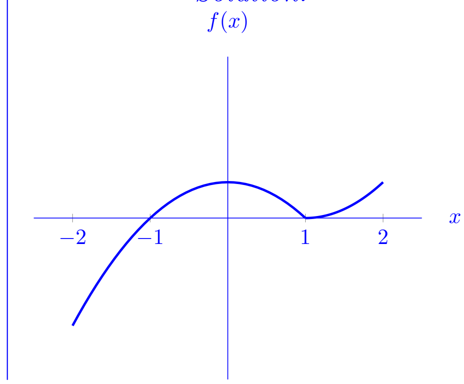
Solution:



b. [3 points]

A continuous function $f(x)$ with domain $[-2, 2]$ that has both a local minimum at $x = 1$ and an inflection point at $x = 1$.

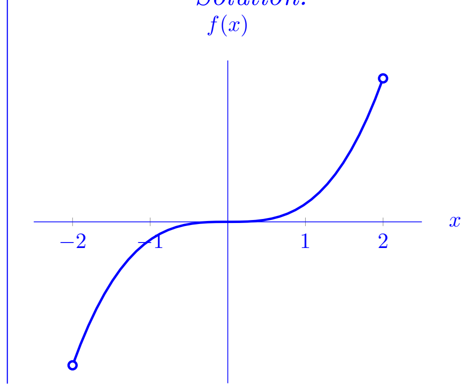
Solution:



c. [3 points]

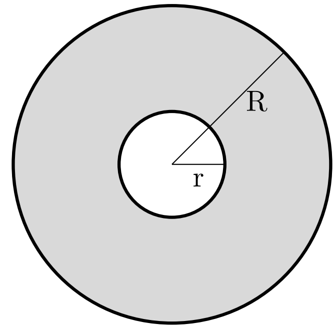
A continuous function $f(x)$ with domain $(-2, 2)$ that has exactly one critical point and no global extrema. Note that this domain differs from those in previous parts.

Solution:



8. [8 points]

Kristen is machining a metal washer to fix her broken down motorcycle. A washer is a flat, circular piece of metal with a hole in the middle. Kristen's washer is depicted by the shaded region in the figure to the right. The washer has an inner radius of r centimeters and an outer radius of R centimeters. The area of the washer must be exactly 5 square centimeters, and r must be at least 1 centimeter.



a. [3 points] Find a formula for r in terms of R.

Solution: The area of the washer is the difference between the outer circle's area and inner circle's area. So, since this must be 5 square centimeters we have $\pi R^2 - \pi r^2 = 5$, so $r^2 = \dfrac{\pi R^2 - 5}{\pi}$, and $r = \sqrt{\dfrac{\pi R^2 - 5}{\pi}}$.

Answer: $r = \sqrt{\dfrac{\pi R^2 - 5}{\pi}}$

b. [2 points] The structural integrity of the washer depends on both its inner radius and its outer radius. Specifically, the structural integrity is given by the equation

$$S = 32R\left(\ln(rR+1) + 7\right).$$

Express S as a function of R. *Your answer should not include r.*

Solution: We substitute our answer from part a. into the formula for S.

Answer: $S(R) = 32R(\ln\left(R\sqrt{\dfrac{\pi R^2 - 5}{\pi}} + 1\right) + 7)$

c. [3 points] What is the domain of $S(R)$ in the context of this problem? You may give your answer as an interval or using inequalities.

Solution: We are told that r must be at least 1. When $r = 1$, we have

$$\pi R^2 - \pi r^2 = 5$$
$$\pi R^2 - \pi = 5$$
$$R^2 = \frac{5+\pi}{\pi}$$
$$R = \sqrt{\frac{5+\pi}{\pi}}.$$

This is the smallest possible value of R, because if we make r larger, R must also be made larger so that the area of the washer can remain 5 square centimeters. There is no upper bound on how large R can be.

Answer: $[\sqrt{\dfrac{5+\pi}{\pi}}, \infty)$

9. [9 points] Consider the function

$$f(x) = \begin{cases} -2e^{2x-2} & x \le 1 \\ x^3 - 3x^2 & x > 1. \end{cases}$$

a. [5 points] Find all critical point(s) of $f(x)$. Write NONE if there are none.

Solution: The derivative of $y = -2e^{2x-2}$ is $\frac{dy}{dx} = -4e^{2x-2}$. So there are no critical points for $x < 1$. Also, at $x = 1$, this piece has a slope of -4.

The derivative of $y = x^3 - 3x^2$ is $\frac{dy}{dx} = 3x^2 - 6x = 3x(x-2)$ which is zero at $x = 0$ and $x = 2$. Only $x = 2$ is on the domain of this piece, so $x = 2$ is a critical point and $x = 0$ is not. Also, at $x = 1$, this piece has a slope of -3.

Since the slopes aren't equal on the left and right sides of $x = 1$, the function $f(x)$ can't be differentiable there. So $x = 1$ is also a critical point.

Answer: Critical point(s) at $x =$ **1 and 2**

b. [4 points] Find the x-coordinate of all global maxima and global minima of $f(x)$ on the interval $(-\infty, 4]$. For each, write NONE if there are none.

Solution: Since $f(x)$ is continuous (because both pieces equal -2 when $x = 1$), we can compare the value of $f(x)$ at the critical points $x = 1$ and $x = 2$ along with the end point $x = 4$, and we need to consider the end behavior as $x \to \infty$:

$f(1) = -2e^{2(1)-2} = -2e^0 = -2$

$f(2) = (2)^3 - 3(2)^2 = 8 - 12 = -4$

$f(4) = (4)^3 - 3(4)^2 = 64 - 48 = 12$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -2e^{2x-2} = 0$

So, $f(x)$ has a global maximum at $x = 4$ and a global minimum at $x = 2$.

Answer: global max(es) at $x =$ 4

Answer: global min(s) at $x =$ 2

10. [8 points] Let $j(t)$ be a differentiable function with domain $(0, \infty)$ that satisfies all of the following:

- $j(5) = 0$
- $j(t)$ has exactly two critical points
- $j(t)$ has a local maximum at $t = 5$
- $j(t)$ has a local minimum at $t = 9$
- $\lim_{t \to 0^+} j(t) = -\infty$
- $\lim_{t \to \infty} j(t) = 0$

You do not need to show work in this problem.

a. [2 points] Circle all of the following intervals on which $j'(t)$ must always be negative.

$(0, 2)$ $\qquad$ $(2, 5)$ $\qquad$ $\boxed{(5, 9)}$ $\qquad$ $(9, \infty)$

b. [3 points] Find all the values of t at which $j(t)$ attains global extrema on the interval $[1, 9]$. If not enough information is provided, write NOT ENOUGH INFO. If there are no such values of t, write NONE.

Solution: We can conclude that $j(t)$ increases until $t = 5$, then decreases until $t = 9$. So, $t = 5$ must be the global max.

By the Extreme Value Theorem, we know $j(t)$ has a global minimum on $[1, 9]$, but we don't know which value, $j(1)$ or $j(9)$, is smaller.

Answer: Global max(es) at $t =$ **5**

Answer: Global min(s) at $t =$ **NOT ENOUGH INFO**

c. [3 points] Find all the values of t at which $j(t)$ attains global extrema on its domain. If not enough information is provided, write NOT ENOUGH INFO. If there are no such values of t, write NONE.

Solution: Similarly to part b., we can conclude that $j(t)$ increases until $t = 5$, then decreases until $t = 9$, then increases again. Since $j(5) = 0$, we know $t = 5$ must be the global max, since $j(t)$ can never again reach 0. (If it did, there would be another critical point, since $\lim_{t \to \infty} j(t) = 0$.)

Also, we know there is no global min since $\lim_{t \to 0^+} j(t) = -\infty$.

Answer: Global max(es) at $t =$ **5**

Answer: Global min(s) at $t =$ **NONE**