Math 115 — Final Exam — April 26, 2019

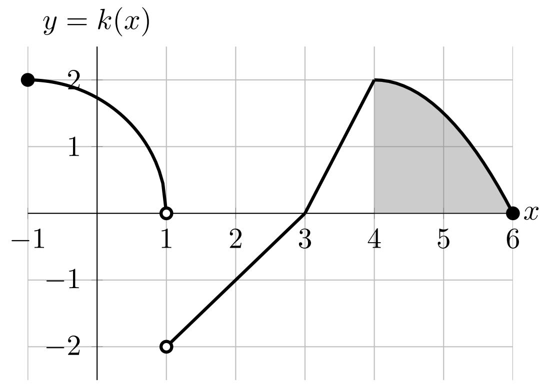
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	8	
3	9	
4	10	
5	8	
6	13	

Problem	Points	Score
7	10	
8	11	
9	8	
10	4	
11	6	
Total	100	

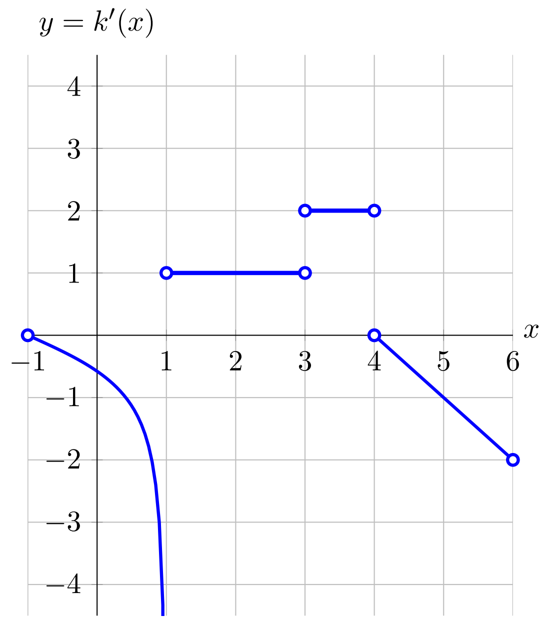
1. [13 points]



A portion of the graph of the function $k(x)$ is shown to the right. Note that:

- $k(x)$ consists of a quarter circle on $-1 \leq x < 1$
- $k(x)$ is piecewise linear on $1 < x \leq 4$
- $k(x) = -\frac{1}{2}(x-4)^2+2$ on the interval $4 \leq x \leq 6$
- the area of the shaded region is $\frac{8}{3}$

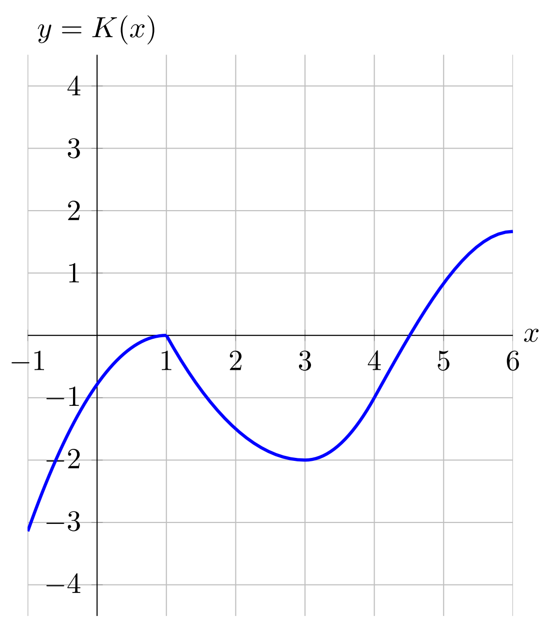
a. [6 points]



On the axes to the right, carefully sketch the graph of $k'(x)$, the derivative of $k(x)$, on the interval $-1 < x < 6$. Be sure that your graph carefully indicates:

- where $k'(x)$ is undefined
- any vertical asymptotes of $k'(x)$
- where $k'(x)$ is zero, positive, and negative
- where $k'(x)$ is increasing, decreasing, and constant
- where $k'(x)$ is linear (with correct slope)

b. [7 points]



Let $K(x)$ be a continuous antiderivative of $k(x)$ with $K(1) = 0$. On the axes to the right, carefully draw a graph of $K(x)$ on $-1 \leq x \leq 6$. Be sure that your graph carefully indicates:

- where $K(x)$ is and is not differentiable
- the values of $K(x)$ at $x = -1, 1, 3, 4,$ and 6
- where $K(x)$ is increasing, decreasing, and constant
- the concavity of $K(x)$ and any inflection points of $K(x)$

2. [8 points] Consider the family of functions

$$g(x) = a\ln(x) + \frac{b}{x},$$

defined for $x > 0$, where a and b are positive constants.

a. [2 points] Any function $g(x)$ in this family has only one critical point. In terms of a and b, what is the x-coordinate of that critical point? Show your work.

Solution: Since

$$g'(x) = \frac{a}{x} - \frac{b}{x^2} = \frac{ax-b}{x^2},$$

we see that $g'(x) = 0$ when $ax - b = 0$, i.e. when $x = \frac{b}{a}$.

Answer: $x =$ b/a

b. [3 points] Is the critical point a local maximum, a local minimum, or neither? Circle your answer below. Use calculus, and be sure to show enough evidence to justify your answer.

Solution: We use the 2nd derivative test. Since

$$g''(x) = -\frac{a}{x^2} + \frac{2b}{x^3} = \frac{-ax+2b}{x^3},$$

we have that

$$g''\left(\frac{b}{a}\right) = \frac{-a\frac{b}{a}+2b}{\left(\frac{b}{a}\right)^3} = \frac{(-b+2b)a^3}{b^3} = \frac{a^3}{b^2} > 0.$$

Answer: local max $\boxed{\text{local min}}$ neither

c. [3 points] Find values of a and b such that $g(x)$ has a critical point at $(e^2, 1)$. Show your work.

Solution: We know that $0 = g'(e^2) = \frac{ae^2-b}{e^4} = 0$, so that $b = ae^2$. We also know that $1 = g(e^2) = a\ln(e^2) + \frac{b}{e^2} = 2a + \frac{ae^2}{e^2} = 3a$, so that $a = 1/3$. Then $b = e^2/3$.

Answer: $a =$ $1/3$ **Answer:** $b =$ $e^2/3$

3. [9 points] Given below is a table of values for an **odd** function $g(x)$, its derivative $g'(x)$, and its second derivative $g''(x)$. The functions $g(x)$, $g'(x)$, and $g''(x)$ are all continuous and defined for all real numbers.

x	0	4	7	9
$g(x)$	0	3	2	7
$g'(x)$	4	-1	0	-3
$g''(x)$	0	6	3	-9

Find the following values exactly, or write NEI if there is not enough information provided to do so. You do not need to show work, but limited partial credit may be awarded for work shown.

a. [2 points] $\displaystyle\lim_{r\to 0}\frac{g'(4+r)+1}{r}$

Solution: This is the limit definition of the derivative of $g'(x)$ at $x=4$. Therefore its value is $g''(4)=6$.

Answer: $=$ **6**

b. [1 point] $g'(-4)$

Solution: The derivative of an odd function is an even function. So, $g'(x)$ is even. This means that $g'(-4)=g'(4)=-1$.

Answer: $=$ -1

c. [2 points] $\displaystyle\int_0^7 \big(g'(x)+1\big)\ dx$

Solution:

$$\begin{aligned} &= \int_0^7 g'(x)\ dx + \int_0^7 1\ dx \\ &= (g(7)-g(0))+7 \\ &= 2+7=9 \end{aligned}$$

Answer: $=$ 9

d. [2 points] $\displaystyle\int_{-8}^8 g(x)\ dx$

Solution: Since $g(x)$ is odd, $\int_{-a}^a g(x)\ dx = 0$ for any choice of a.

Answer: $=$ **0**

e. [2 points] the average value of $g''(x)$ on $[7,9]$

Solution:

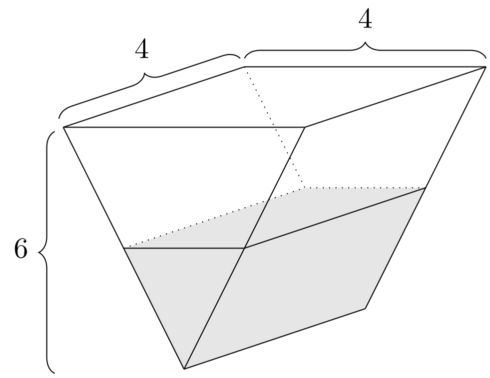
$$\begin{aligned} \frac{1}{2}\int_7^9 g''(x)\ dx &= \frac{1}{2}\big(g'(9)-g'(7)\big) \\ &= \frac{-3}{2} \end{aligned}$$

Answer: $=$ $\frac{-3}{2}$

4. [10 points]

a. [5 points]

Sam is pouring concrete into a hole in the shape of a triangular prism. The hole is 4 meters wide, 4 meters long, and 6 meters deep at its deepest point. A partially filled hole with the correct dimensions is shown to the right.



Sam is looking down into the hole and observes that the rectangular top surface of the concrete is growing at a rate of 0.8 meters squared per minute. Find the rate at which the depth of the concrete is growing. *Include units.*

Solution: Let w denote the width of the surface rectangle and h denote the depth of the concrete. By similar triangles, $w = \frac{2}{3}h$. Then the area, $A(h)$, is

$$A(h) = 4\left(\frac{2}{3}h\right) = \frac{8}{3}h.$$

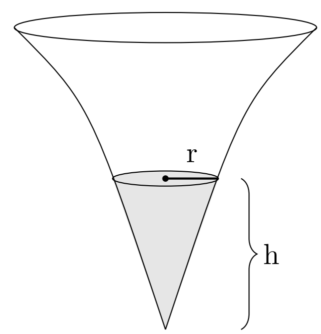
Taking the derivative with respect to time we have $\frac{dA}{dt} = \frac{8}{3}\frac{dh}{dt}$, so $\frac{dh}{dt} = 0.8\left(\frac{3}{8}\right) = 0.3$.

Answer: 0.3 **meters per minute**

b. [5 points]

Donna is pouring concrete into a different hole, which is in the shape of a horn as shown to the right. When the concrete has been poured to a depth of h meters and its surface has radius r, the volume of the poured concrete is given by

$$V = \frac{\pi}{7}r^2h.$$



When the depth of the concrete that has been poured is 0.8 meters, the radius of its surface is 0.5 meters, the radius is growing at a rate of 5 meters per hour, and the volume is growing at a rate of 2 cubic meters per hour. How fast is the depth changing? *Include units.*

Solution:

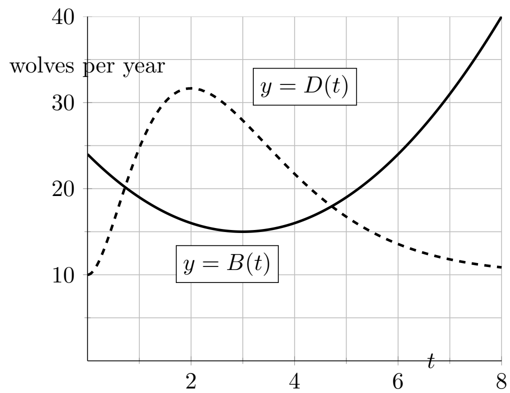
$$\frac{dV}{dt} = \frac{\pi}{7}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$$
$$2 = \frac{\pi}{7}\left(2(0.5)(0.8)(5) + (.5)^2\frac{dh}{dt}\right)$$
$$\frac{14}{\pi} = 4 + 0.25\frac{dh}{dt}$$
$$\frac{14}{\pi} - 4 = 0.25\frac{dh}{dt}$$
$$4\left(\frac{14}{\pi} - 4\right) = \frac{dh}{dt}$$

Answer: $4\left(\frac{14}{\pi} - 4\right) \approx 1.825$ **meters per hour**

5. [8 points] The wolf population in the Upper Peninsula of Michigan (UP) has been closely monitored since recovering from near extinction several decades ago.

- Let $B(t)$ be the rate, in wolves per year, at which wolves were being born in the UP t years after January 1st, 2010.
- Let $D(t)$ be the rate, in wolves per year, at which wolves were dying in the UP t years after January 1st, 2010.

Suppose that the graphs of $B(t)$ (solid) and $D(t)$ (dashed) are as shown below.



Assume that no wolves migrate in and out of the UP. That is, any wolf born in the UP remains there, and no wolves born elsewhere travel to the UP. Also, let W_0 denote the number of wolves in the UP on January 1st, 2010.

In parts **a.** and **b.**, your mathematical expressions may involve $B(t)$, $D(t)$, their derivatives, W_0, and/or definite integrals.

a. [2 points] Write an expression that represents the total number of wolves born in the UP in the year 2017.

Answer: $\int_7^8 B(t)\ dt$

b. [3 points] Write an expression that represents the total number of wolves living in the UP on January 1st, 2018.

Answer: $W_0 + \int_0^8 B(t) - D(t)\ dt$

c. [1 point] Is there a time $t > 0$ when the number of wolves in the UP was equal to W_0? If so, estimate the first such time; if not, write NONE.

Answer: $t \approx$ 1.3

d. [2 points] Estimate the time(s) t, for $0 \leq t \leq 8$, when the population of wolves in the UP was the smallest.

Answer: $t \approx$ 4.7

6. [13 points] The following are tables of values for two differentiable functions $f(x)$ and $g(x)$ and their derivatives. Missing values are denoted by a "?". Assume that each of these functions is defined for all real numbers, that $f'(x)$ and $g'(x)$ are continuous, and that $g(x)$ is invertible.

x	0	2	3	6	9
$f(x)$	-1	?	0	-2	?
$f'(x)$	1	4	-1	?	1

x	-1	1	3	7	11
$g(x)$	-4	1	2	6	7
$g'(x)$	7	?	3	4	?

a. [4 points] For each of the following, find the value exactly. If there is not enough information to find the quantity, write NEI.

i. [2 points] Let $z(x) = f(g(x))$. Find $z'(3)$.

Answer: $z'(3) =$ **12**

ii. [2 points] Let $j(x) = g^{-1}(x)$. Find $j'(7)$.

Answer: $j'(7) =$ **NEI**

b. [2 points] Use a left-hand Riemann sum with three equal subintervals to estimate $\int_{-1}^{11} g(x)\, dx$. Write out all the terms in your sum.

Answer: $4(-4+2+6)$

c. [1 point] Is your answer in part **b.** an overestimate or an underestimate? Circle your answer. If there is not enough information circle NEI.

Answer: OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ NEI

d. [4 points] The function $f(x)$ has two critical points, at $x = 2.5$ and $x = \pi$. These are the only critical points of $f(x)$. For each critical point, decide if it is a local max, local min, neither, or if there is not enough information to determine this (NEI). Circle your answers.

Answer: $x = 2.5$ is a: LOCAL MIN $\boxed{\text{LOCAL MAX}}$ NEITHER NEI

Answer: $x = \pi$ is a: $\boxed{\text{LOCAL MIN}}$ LOCAL MAX NEITHER NEI

e. [2 points] On which of the following interval(s) *must* $f(x)$ have an inflection point? Circle all correct answers.

$[0, 3]$ $[2, 3]$ $[3, 9]$

Solution: We know that for $f(x)$ to have an inflection point p, the sign of $f''(x)$ must change at p. The sign of $f''(x)$ must change during the interval $[0, 3]$, but it does not happen at a single point – for example, this can occur if $f''(x)$ is first positive, and then zero over an interval, and then negative. So the correct answer is that *none* of the intervals must have an inflection point; however full credit was also given if only $[0, 3]$ was circled.

7. [10 points] Zerina owns a small business selling custom screen-printed and embroidered apparel.

a. Zerina receives orders for embroidered polo shirts, which she sells for \$11 each. The cost, in dollars, for her to complete an order of q embroidered polo shirts is

$$C(q) = \begin{cases} 6q - \frac{1}{8}q^2 + \frac{56}{9} & 0 \le q \le 16 \\ \frac{2}{9}q^{3/2} + 10q - 104 & q > 16. \end{cases}$$

Note that $C(q)$ is continuous for all $q \ge 0$.

i. [1 point] What is the fixed cost, in dollars, of an order of embroidered polo shirts?

Answer: 56/9

ii. [5 points] Find the quantity q of embroidered polo shirts in an order that would result in the most profit for Zerina. Assume that, because of storage constraints, Zerina cannot accept an order for more than 80 embroidered polo shirts. Use calculus to find and justify your answer, and make sure you provide enough evidence to fully justify your answer.

Solution: We are given $C(q)$, and know that $R(q) = 11q$. Then since $\pi(q) = R(q) - C(q)$, any point at which $MR = MC$ is a critical point of $\pi(q)$. Now $MR(q) = 11$ and

$$MC(q) = \begin{cases} 6 - \frac{1}{4}q & 0 \le q < 16 \\ \frac{1}{3}q^{1/2} + 10 & q > 16. \end{cases}$$

We set $MR = MC$ in both of these cases:

$$6 - \frac{1}{4}q = 11 \qquad\qquad \frac{1}{3}q^{1/2} + 10 = 11$$

$$-\frac{1}{4}q = 5 \qquad\qquad q^{1/2} = 3$$

$$q = -20 \qquad\qquad q = 9$$

but neither critical point falls within the domain of the appropriate formula. So there are no points at which $MR = MC$. However, MC is undefined at $q = 16$, since if we plug 16 in to both pieces of $MC(q)$ we get different values. Therefore $\pi'(q)$ is also undefined at $q = 16$. This is the only critical point.

So the possible locations for the global maximum are the endpoints 0 and 80 and the critical point 16. Since $\pi(0) = -56/9$, $\pi(80) \approx 25$, and $\pi(16) \approx 105.7$, an order of 16 polo shirts would result in the most profit for Zerina.

Answer: $q =$ **16**

b. [3 points] Zerina also receives orders for screen-printed t-shirts. When a customer places such an order, they pay a \$6 setup fee, plus \$9 per t-shirt for the first 20 t-shirts ordered. Any additional t-shirts ordered only cost \$7 per t-shirt. Let $P(s)$ be the total price, in dollars, a customer pays for an order of s screen-printed t-shirts. Find a formula for $P(s)$.

Answer: $$P(s) = \begin{cases} 6 + 9s & \text{if } 0 \le s \le 20 \\ 186 + 7(s - 20) & \text{if } s > 20 \end{cases}$$

8. [11 points] The energy, in megajoules (MJ), produced by a wind turbine depends on the speed of the wind. In particular, suppose $P(s)$ is the power, in megajoules per hour (MJ/h), produced by the turbine when the speed of the wind is s kilometers per hour (km/h). Also suppose that $W(t)$ gives the wind speed, in km/h, at the turbine's location t hours after noon on a typical day.

Assume that $P(s)$ is invertible, and that both $P(s)$ and $W(t)$ are differentiable.

a. [2 points] Give a practical interpretation of the equation $P(W(0)) = 8$.

Solution: At noon on a typical day, the turbine produces 8 MJ/h of power.

b. [3 points] Give a practical interpretation of the equation $\int_0^5 P(W(t))\, dt = 46$.

Solution: From noon to 5 p.m. on a typical day, the turbine generates 46 MJ of energy.

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$W'(4) = 21$$

From 4 pm to 4:10 pm, ...

Solution: the wind speed at the turbine's location increases by approximately 3.5 km/h.

d. [3 points] Circle the one statement below that is best supported by the equation

$$(P^{-1})'(13) = 2.9.$$

i. *If the turbine is producing 13 MJ/h of power, the wind speed must increase by approximately 2.9 km/h to produce an additional MJ/h of power.*

ii. If the wind is blowing at 13 km/h and increases to 14 km/h, the power produced by the turbine will increase by about 2.9 MJ/h.

iii. If the wind speed is 13 km/h, the power generation of the turbine will increase by one MJ/h if the wind speed increases to about 15.9 km/h.

iv. When the turbine is generating 13 MJ/h of power, an increase of one km/h in wind speed will produce approximately 2.9 MJ/h more power.

9. [8 points] A home improvement store is designing a new bucket to sell. The bucket will be in the shape of a cylinder, so that the volume V of the bucket is given by

$$V = \pi r^2 h,$$

where r is the bucket's radius and h is the bucket's height, both measured in feet. Since the bucket does not have a top, the surface area A of the bucket is given by

$$A = 2\pi r h + \pi r^2.$$

The store has decided that the new bucket should have a volume of exactly 1 cubic foot. Find the dimensions of the bucket that will minimize its surface area. Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact minimize the surface area.

Solution: Since we want the bucket to have a volume of 1, we have $1 = \pi r^2 h$, or $h = \frac{1}{\pi r^2}$. Thus

$$A = 2\pi r\left(\frac{1}{\pi r^2}\right) + \pi r^2 = \frac{2}{r} + \pi r^2,$$

and

$$\frac{dA}{dr} = -\frac{2}{r^2} + 2\pi r.$$

Setting this derivative equal to 0, we find that $2\pi r = \frac{2}{r^2}$, so $\pi r^3 = 1$, and $r = \frac{1}{\pi^{1/3}}$.
To see that this is the global minimum, we use the 2nd derivative test:

$$\frac{d^2A}{dr^2} = \frac{4}{r^3} + 2\pi > 0,$$

so $r = \frac{1}{\pi^{1/3}}$ is a local minimum. Since this was the only critical point on the domain of $(0, \infty)$, it must be the global minimum. Finally, when $r = \frac{1}{\pi^{1/3}} = \pi^{-1/3}$, we see that $h = \frac{1}{\pi\pi^{-2/3}} = \frac{1}{\pi^{1/3}}$.

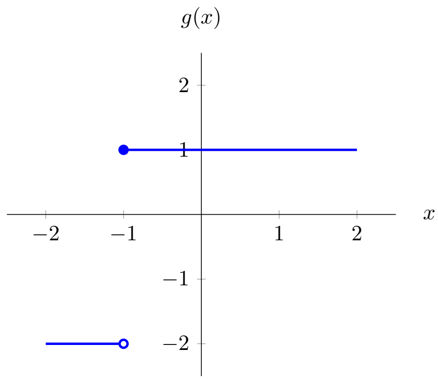
Answer: surface area is minimized when $r = \frac{1}{\pi^{1/3}}$ $h = \frac{1}{\pi^{1/3}}$

10. [4 points] For each part, draw a function on the given axes that satisfies the given conditions. Or, if no such function exists, write DNE. Make sure your graphs are clear and unambiguous.

Solution: Note that for both graphs, there are many functions that satisfy the listed properties.

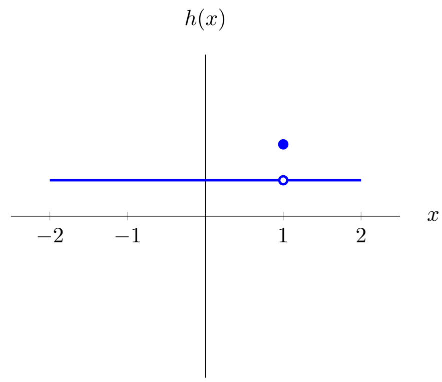
a. [2 points]
A function $g(x)$ that satisfies

- $\lim_{x\to -1^+} g(x) = 1$ and
- $\lim_{x\to -1^-} g(x) = -2$.



b. [2 points]
A function $h(x)$ that satisfies

- $\lim_{x\to a} h(x)$ exists for every $-2 < a < 2$ and
- $h(x)$ is not continuous at $x = 1$.



11. [6 points]
Suppose that $T(x) = A\cos\left(\frac{\pi}{2}x\right) + C$, where A and C are constants. To the right is a table of values for $T(x)$.

x	0	2	3
$T(x)$	10	-2	4

a. [1 point] What is the period of $T(x)$?

Solution: We know we can find the period using $\frac{2\pi}{B}$, and for $T(x)$ we see that $B = \frac{\pi}{2}$. So, the period is $\frac{2\pi}{\pi/2} = \frac{4\pi}{\pi} = 4$.

Answer: period = **4**

b. [2 points] Find the values of A and C.

Solution: The amplitude A is half the difference between the largest and smallest values of $T(x)$. Since the period is 4 and there is no horizontal shift, the largest value of $T(x)$ occurs at $x = 0$ and the smallest value occurs at $x = 2$. So the amplitude is $\frac{1}{2}(T(0) - T(2)) = 12$.

The vertical shift C is the midpoint value of $T(x)$. Since the period is 4 and there is no horizontal shift, this occurs at $x = 1$ and $x = 3$. From the table we see that $T(3) = 4$.

Answer: $A =$ **6** **Answer:** $C =$ **4**

c. [3 points] Let $Q(x)$ be the quadratic approximation of $T(x)$ at $x = 2$. Find a formula for $Q(x)$. Your answer should not include the constants A or C.

Answer: $Q(x) = -2 + \frac{3\pi^2}{4}(x-2)^2$