Math 115 — First Midterm — February 11, 2020

EXAM SOLUTIONS

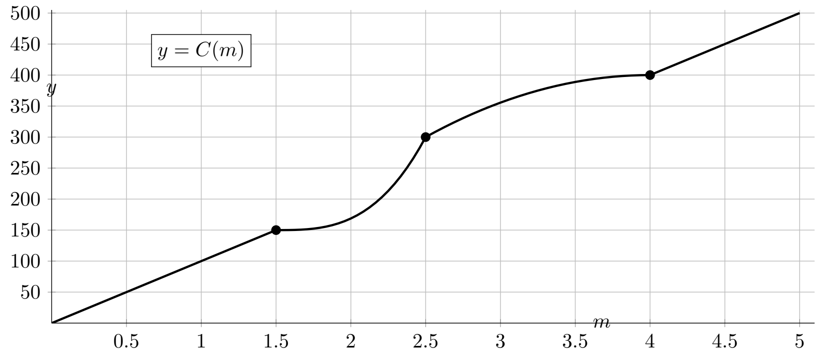
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	5	
3	11	
4	10	
5	9	

Problem	Points	Score
6	14	
7	11	
8	11	
9	9	
10	9	
Total	100	

1. [11 points] Reiner recently went for a 5-mile run. Let $R(t)$ be Reiner's distance, in miles, t minutes after he started his run, and let $C(m)$ be the number of calories that Reiner had burned after running m miles. A table giving some values of $R(t)$ and a graph of $C(m)$ are given below. Assume that the functions are invertible, and note that $C(m)$ is linear for $0 < m < 1.5$ and $4 < m < 5$.

t	0	6	10	16	23	27	32	34
$R(t)$	0	0.8	1.3	2.5	3.2	3.8	4	4.4



a. [5 points] Compute the following quantities **exactly**. If the quantity does not exist, write DNE, or if there is not enough information to compute it exactly, write NEI.

i. [1 points] How many minutes does it take for Reiner to run his first 4 miles?

Answer: = $R^{-1}(4) = 32$

ii. [2 points] How many calories has Reiner burned after running for 10 minutes?

Answer: = $C(R(10)) = 130$

iii. [2 points] How many minutes does it take for Reiner to burn his first 300 calories?

Answer: = $R^{-1}(C^{-1}(300)) = 16$

b. [2 points] Compute the average rate of change of $C(m)$ from $m = 1.5$ to $m = 4$. Include units.

Answer: = 100 **calories per mile**

c. [2 points] Estimate $C'(\pi)$. Include units.

Answer: = ≈ 66.7 **calories per mile**

d. [2 points] Estimate Reiner's instantaneous velocity 34 minutes into his run. Include units.

Answer: = ≈ 0.2 **miles per minute**

2. [5 points] Let

$$Q(r) = 1 + r^{\ln(r)}.$$

Use the limit definition of the derivative to write an explicit expression for $Q'(5)$. *Your answer should not involve the letter Q. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Answer: $Q'(5) = \lim_{h\to 0} \dfrac{1+(5+h)^{\ln(5+h)} - \left(1+5^{\ln(5)}\right)}{h}$

3. [11 points] Inga, a beekeeper, sets up a new hive on April 1. At two later times, she estimates the hive's population. These estimates are shown in the table below.

weeks after April 1	2	5
population of the hive, in thousands	7.7	10.9

a. [2 points] Find a formula for a linear function $L(t)$ modeling the hive's population, in thousands, t weeks after April 1.

Answer: $L(t) = \frac{16}{15}(t-2) + 7.7$

b. [4 points] Find a formula for an exponential function $E(t)$ modeling the the hive's population, in thousands, t weeks after April 1.

Solution: Since $E(2) = 7.7$ and $E(5) = 10.9$ and we know $E(t) = ab^t$ for some a and b,

$$10.9 = ab^5$$
$$7.7 = ab^2$$
$$\frac{10.9}{7.7} = b^3$$
$$b = \left(\frac{10.9}{7.7}\right)^{1/3} \approx 1.1228 \text{ and } a = \frac{7.7}{b^2} = \frac{7.7}{\left(\frac{10.9}{7.7}\right)^{2/3}} \approx 6.1076.$$

Then $E(t) = \dfrac{7.7}{\left(\frac{10.9}{7.7}\right)^{2/3}}\left(\dfrac{10.9}{7.7}\right)^{t/3} \approx 6.1076(1.1228)^t.$

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Inga is now studying the populations of two other hives. She determines that the population, in thousands, t weeks after April 1 of her hive of Carniolan bees can be modeled by

$$C(t) = 9e^{0.14t},$$

while the population, in thousands, of her hive of Starline bees can be modeled by

$$S(t) = 17(1.05)^t.$$

c. [1 point] By what percent is the hive of Carniolan bees growing each week?

Solution: Since $e^{0.14} \approx 1.01502$, we have that the growth rate is $e^{0.14} - 1 \approx 0.0152$ or 15.02%.

Answer: 15.02 %

d. [4 points] At what time t will the populations of these two hives be equal? Give your answer in **exact form**, and show every step of your algebraic work.

Solution:

Method 1:

$$9e^{0.14t} = 17(1.05)^t$$
$$\ln\left(9e^{0.14t}\right) = \ln\left(17(1.05)^t\right)$$
$$\ln(9) + \ln\left(e^{0.14t}\right) = \ln(17) + \ln\left(1.05^t\right)$$
$$\ln(9) + 0.14t = \ln(17) + t\ln(1.05)$$
$$\ln(9) - \ln(17) = t(\ln(1.05) - 0.14)$$
$$t = \frac{\ln(9) - \ln(17)}{\ln(1.05) - 0.14}$$

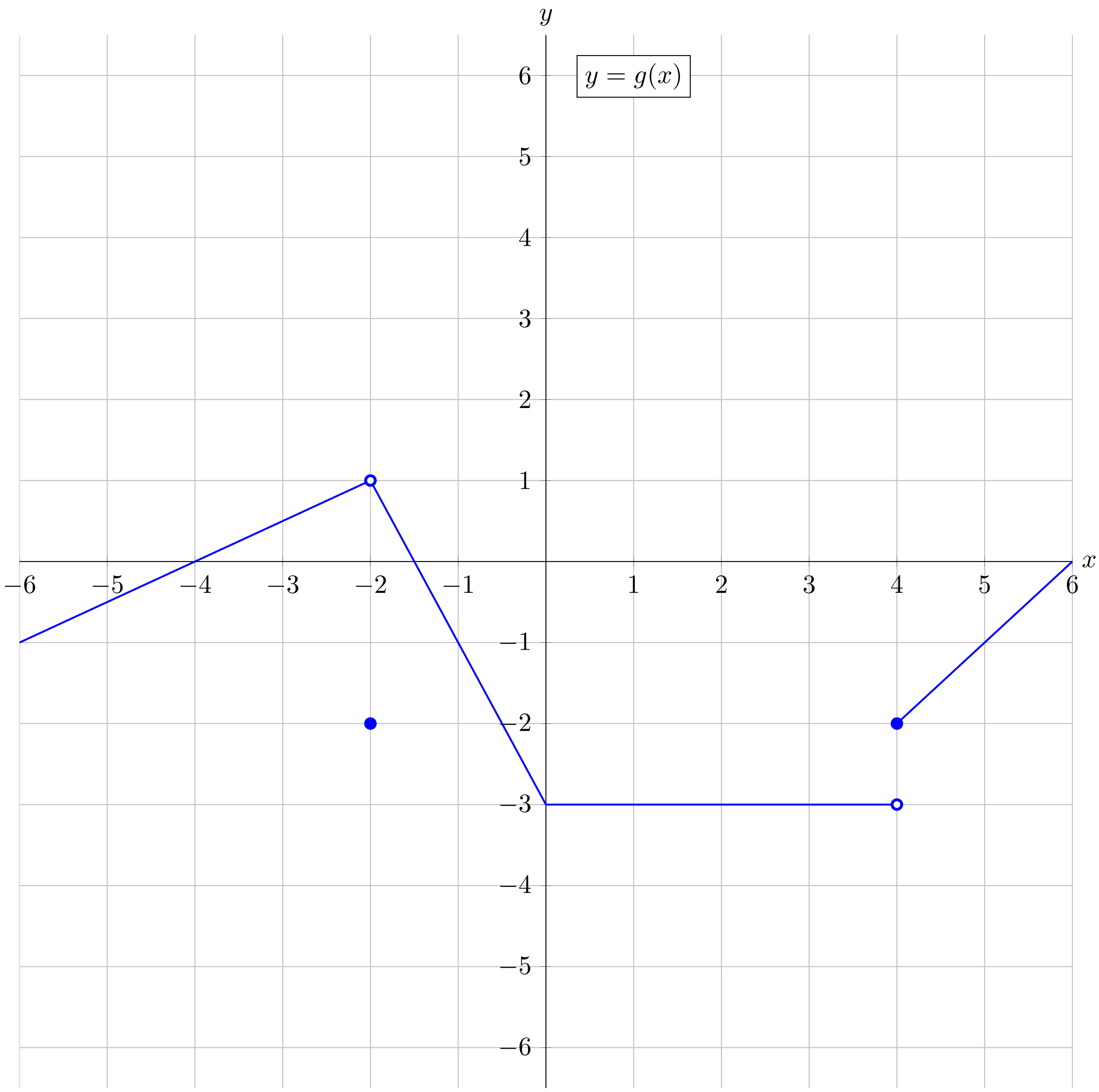
Method 2:

$$9e^{0.14t} = 17(1.05)^t$$
$$\frac{9}{17} = \frac{1.05^t}{e^{0.14t}}$$
$$\frac{9}{17} = \left(\frac{1.05}{e^{0.14}}\right)^t$$
$$\ln\left(\frac{9}{17}\right) = \ln\left(\left(\frac{1.05}{e^{0.14}}\right)^t\right)$$
$$\ln\left(\frac{9}{17}\right) = t\ln\left(\frac{1.05}{e^{0.14}}\right)$$
$$t = \frac{\ln\left(\frac{9}{17}\right)}{\ln\left(\frac{1.05}{e^{0.14}}\right)}$$

Answer: $\frac{\ln(9) - \ln(17)}{\ln(1.05) - 0.14}$

4. [10 points] On the axes provided below, sketch the graph of a single function $g(x)$ that satisfies all of the following conditions:

- the domain of the function $g(x)$ contains $-6 < x < 6$
- $g(x)$ is increasing for $-5 < x < -2$
- $\lim_{x\to -2} g(x) = 1$
- $g(x)$ is <u>not</u> continuous at -2
- $g(0) = -3$
- the average rate of change of $g(x)$ from $x = -2$ to $x = 0$ is $-\frac{1}{2}$
- $g(x)$ is constant for $0 < x < 3$
- $\lim_{x\to 4^-} g(x) = g(4)$
- $g(x)$ is <u>not</u> continuous at 4
- $g'(x)$ is constant for $4 < x < 6$



5. [9 points]

a. [4 points] Each year, a lake reaches its maximum temperature of 76 degrees Fahrenheit (°F) on September 1, and its minimum temperature of 40°F on March 1. Write a formula for a sinusoidal function $T(m)$ modeling the lake's temperature, in °F, m months after January 1.

Solution: We know that the amplitude must be half of $76 - 40 = 36$, or 18, and that the vertical shift must be $40 + 18 = 58$. We also see that the period is 12 months, and that a minimum occurs on March 1, which is two months after January 1.

Answer: $T(m) =$ $-18\cos\left(\frac{\pi}{6}(m-2)\right) + 58$

b. [5 points] The depth $D(m)$ of this lake, in feet, can also be represented by a sinusoidal function, namely

$$D(m) = 985 + 20\sin\left(\frac{\pi}{6}m\right),$$

where m is the time in months after January 1. Find the amount of time, in months, each year when the depth of the lake is at least 1000 feet. Give your answer in **exact form**.

Solution: We set the equation equal to 1000 and solve:

$$985 + 20\sin\left(\frac{\pi}{6}m\right) = 1000$$

$$\sin\left(\frac{\pi}{6}m\right) = \frac{3}{4}, \quad \text{so that one possible solution is}$$

$$\frac{\pi}{6}m = \arcsin\left(\frac{3}{4}\right)$$

$$m = \frac{6}{\pi}\arcsin\left(\frac{3}{4}\right) \quad \text{which is } \approx 1.6197.$$

The next solution, which could also be found using symmetries of the graph of $D(m)$, is

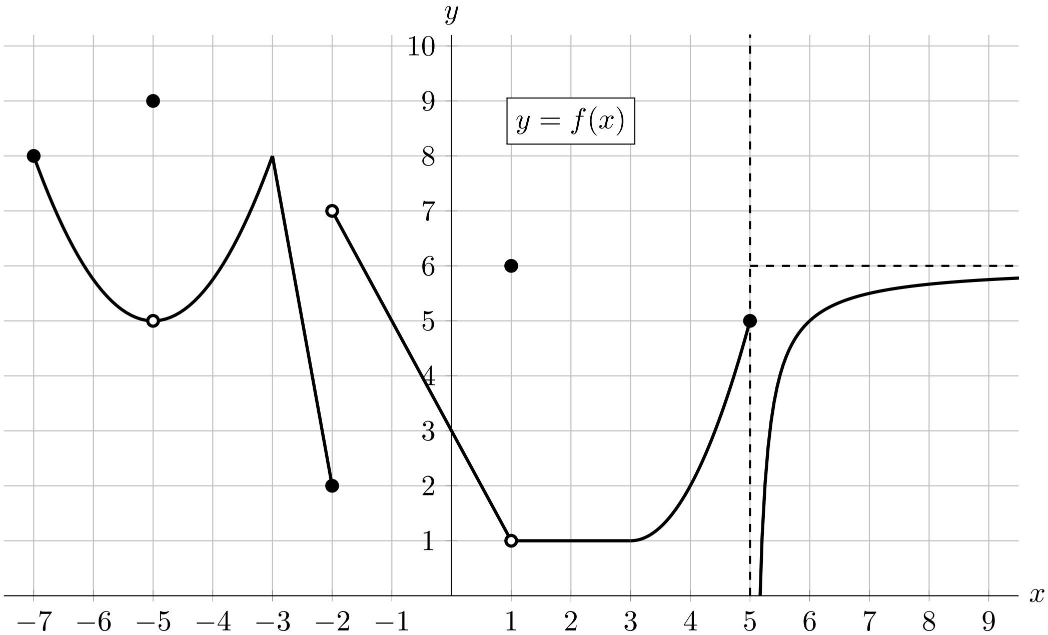
$$\frac{\pi}{6}m = \pi - \arcsin\left(\frac{3}{4}\right)$$

$$m = \frac{6}{\pi}\left(\pi - \arcsin\left(\frac{3}{4}\right)\right) \quad \text{which is } \approx 4.3803.$$

Thus our final answer is $\frac{6}{\pi}\left(\pi - \arcsin\left(\frac{3}{4}\right)\right) - \frac{6}{\pi}\arcsin\left(\frac{3}{4}\right) = 6 - \frac{12}{\pi}\arcsin\left(\frac{3}{4}\right)$.

Answer: $6 - \frac{12}{\pi}\arcsin\left(\frac{3}{4}\right)$

6. [14 points] Below is a portion of the graph of a function $f(x)$ with domain $[-7, \infty)$. Note that $f(x)$ is linear for $-3 < x < -2$ and $-2 < x < 1$, and that $f(x)$ has a vertical asymptote of $x = 5$ and a horizontal asymptote of $y = 6$.



Evaluate each of the following quantities. If a limit diverges to ∞ or $-\infty$ or if the limit does not exist for any other reason, write DNE. You do not need to show work in this problem.

a. [2 points] $\lim_{x\to -2^+} f(x)$

Answer: $=$ 7

b. [2 points] $\lim_{x\to -5} f(x)$

Answer: $=$ 5

c. [2 points] $\lim_{h\to 0} \dfrac{f(h) - f(0)}{h}$

Answer: $=$ -2

d. [2 points] $\lim_{x\to -5^-} f(-x)$

Answer: $=$ DNE

e. [2 points] $\lim_{x\to 2} f(f(x))$

Answer: $=$ 6

Define the function $g(x) = \dfrac{1}{3} f(2x) + 7$. Fill in the blanks below.

f. [2 points] The function $g(x)$ has a horizontal asymptote of $y =$ 9 .

g. [2 points] The function $g(x)$ has a vertical asymptote of $x =$ 5/2 .

7. [11 points] Falco decides to raise sheep and cows on his farm.

- Let $W(p)$ be the amount of wool, in pounds per year, produced by a sheep who was fed p pounds of food per day.
- Let $M(p)$ be the amount of milk, in thousands of gallons per year, produced by a cow who was fed p pounds of food per day.

The functions $W(p)$ and $M(p)$ are differentiable and invertible.

a. [2 points] Use a complete sentence to give a practical interpretation of the the equation

$$W^{-1}(28) = 10.$$

Solution: A sheep fed 10 pounds of food per day produced 28 pounds of wool per year.

b. [3 points] Write a single equation representing the following statement in terms of the functions W, M, and/or their inverses:

A sheep that produced 12 pounds of wool per year was fed 5 fewer pounds of food per day than a cow that produced 2430 gallons of milk per year.

Answer: $M^{-1}(2.43) = W^{-1}(12) + 5$

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$M'(23) = 0.15.$$

If Falco feeds a cow 22.4 pounds of food per day instead of 23 pounds of food per day, then ...

Solution: ...the cow will produce approximately 90 fewer gallons of milk per year.

d. [3 points] Circle the one sentence that gives a valid interpretation of the equation

$$(W^{-1})'(12) = 0.7.$$

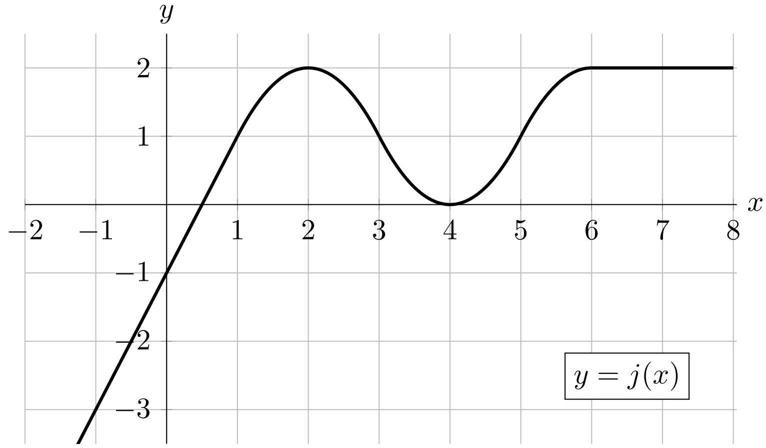
(i.) To increase a sheep's wool production from 12 pounds per year to 12.6 pounds per year, Falco should feed it about 0.42 extra pounds of food per day.

ii. To increase a sheep's wool production from 12 pounds per year to 12.1 pounds per year, Falco should feed it approximately 0.7 more pounds of food per day.

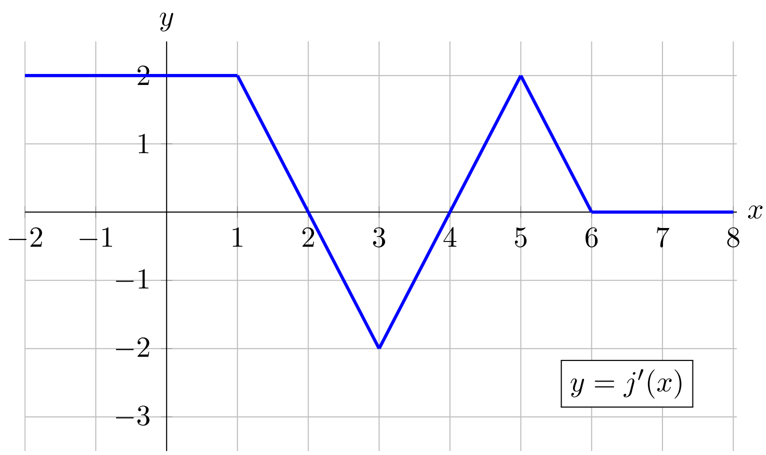
iii. A sheep that is fed 12.3 pounds of food per day instead of 12 pounds of food per day will produce approximately 0.21 additional pounds of wool per year.

iv. When a sheep produces 12 pounds of wool per year, feeding it an extra pound of food per day will cause it to produce about 0.7 additional pounds of wool per year.

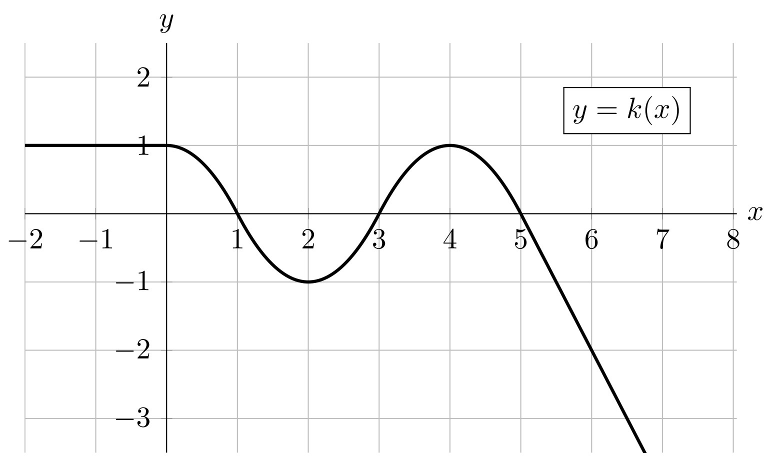
8. [11 points] A portion of the graph of the function $j(x)$ is shown below. Note that $j(x)$ is linear for $x < 1$ and $x > 6$.



a. [7 points] On the axes below, carefully sketch the graph of $j'(x)$, the derivative of $j(x)$, on the interval $-2 < x < 8$. Be sure that your graph carefully indicates where $j'(x)$ is zero, positive, and negative, and where $j'(x)$ is increasing, decreasing, and constant.



b. [4 points] Shown below is a portion of the graph of a function $k(x)$ which can be obtained from $j(x)$ through one or more graph transformations. Find a formula for $k(x)$ in terms of $j(x)$.



Answer: $k(x) =$ $j(-(x-6)) - 1$

9. [9 points] You do not need to show work in this problem, but limited partial credit may be awarded for work shown.

a. [5 points] Consider the rational function

$$g(x) = \frac{(Bx^A + 7)(4x - C)}{(3x^2 + 5)(2x - 12)(x - D)},$$

where A, B, C, and D are constants. Suppose that

- $y = 8$ is a horizontal asymptote of $g(x)$
- $x = 5$ is the only vertical asymptote of $g(x)$.

Find the values of A, B, C, and D.

Solution:
For a non-zero horizontal asymptote to exist, the degree of the numerator and denominator must be equal. Since the denominator has degree 4, we must have $A = 3$.

The horizontal asymptote is then equal $\frac{4B}{6}$, so for this to equal 8, we must have $B = 12$.

A vertical asymptote at $x = 5$ tells us the denominator must be zero at $x = 5$, so $D = 5$.

Since there is only one vertical asymptote, we know $x = 6$ must be a hole. Thus $4x - C = 0$ when $x = 6$, so $C = 24$.

Answer: $A =$ **3** $B =$ **12** $C =$ **24** $D =$ **5**

b. [4 points] Consider the piecewise function

$$h(x) = \begin{cases} E + \dfrac{28}{3^x + 4} & x \leq 1 \\ G + \dfrac{F}{7^x + 5} & x > 1 \end{cases}$$

where E, F, and G are constants. Suppose that

- $\lim_{x \to \infty} h(x) = 8.5$
- $\lim_{x \to -\infty} h(x) = 12$
- $h(x)$ is continuous at $x = 1$.

Find the values of E, F, and G.

Solution:
As x approaches infinity, $\frac{F}{7^x+5}$ goes to 0, so $G = 8.5$.

As x approaches negative infinity, $\frac{28}{3^x+4}$ goes to $\frac{28}{4} = 7$. Thus $E + 7 = 12$, so $E = 5$.

Since we know $h(x)$ is continuous at 1, we must have

$$E + \frac{28}{3+4} = G + \frac{F}{7^x + 5},$$

which reduces to $9 = 8.5 + \frac{F}{12}$. This gives us $F = 6$.

Answer: $E =$ **5** $F =$ **6** $G =$ **8.5**

10. [9 points] Let $P(t)$ be a town's population, in thousands of people, t years after the beginning of 2000. Some values of $P'(t)$, the **derivative** of $P(t)$, are given in the table below.

t	-8	-3	0	3	6	8	12	15
$P'(t)$	2	2	0	0	3	0	-6	-2

Assume that between each pair of consecutive values of t given in the table, $P'(t)$ is either **always increasing**, **always decreasing**, or **always constant**.

a. [1 point] Let $y = P'(t)$. What are the units of y?

Answer: = thousands of people per year

For each of the following, circle **all** correct answers.

b. [2 points] At which of the following time(s) is the town's population increasing?

[$t = -6$] $\quad t = 2 \quad$ [$t = 7$] $\quad t = 13 \quad$ NONE OF THESE

c. [2 points] On which of the following interval(s) is the town's population constant?

$(-7, -5) \quad$ [$(1, 2)$] $\quad (7, 10) \quad$ NONE OF THESE

d. [2 points] On which of the following interval(s) is $P(t)$ linear?

[$(-7, -5)$] $\quad$ [$(1, 2)$] $\quad (7, 10) \quad$ NONE OF THESE

e. [2 points] At which of the following time(s) is the town's population the largest?

$t = 3 \quad t = 6 \quad$ [$t = 8$] $\quad t = 15$