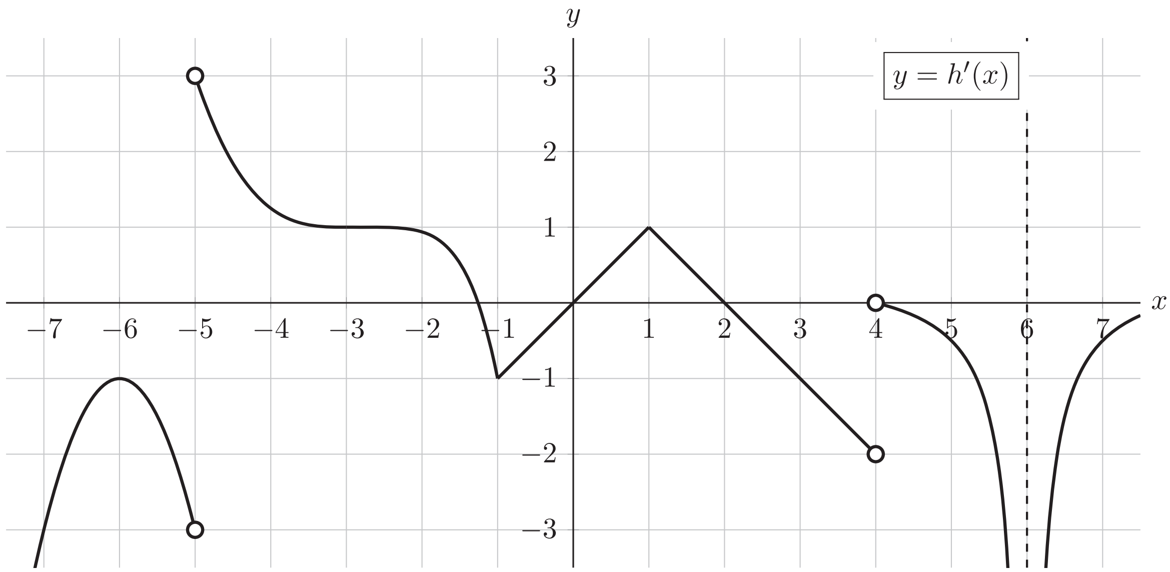
Note: exam problem numbering is off by 1

2. [12 points] A function $h(x)$ is defined and continuous on $(-\infty, \infty)$. A portion of the graph of $h'(x)$, **the derivative of** $h(x)$, is shown below. Note that $h'(x)$ has a vertical asymptote at $x = 6$.



In each part **a.–f.** below, select **all** correct choices.

a. [2 points] At which of the following value(s) does $h(x)$ have a critical point?

$x = -6$ $\qquad$ $x = -3$ $\qquad$ $\boxed{x = 0}$ $\qquad$ $x = 1$ $\qquad$ NONE OF THESE

b. [2 points] At which of the following value(s) does $h(x)$ have a local minimum?

$\boxed{x = -5}$ $\qquad$ $x = -1$ $\qquad$ $x = 2$ $\qquad$ $x = 6$ $\qquad$ NONE OF THESE

c. [2 points] At which of the following value(s) does $h(x)$ have an inflection point?

$\boxed{x = -6}$ $\qquad$ $x = -5$ $\qquad$ $x = -3$ $\qquad$ $\boxed{x = 6}$ $\qquad$ NONE OF THESE

d. [2 points] On which of the following interval(s) is $h(x)$ increasing on the entire interval?

$\boxed{(-5, -3)}$ $\qquad$ $(-1, 1)$ $\qquad$ $(6, 7)$ $\qquad$ NONE OF THESE

e. [2 points] On which of the following interval(s) is $h(x)$ concave down on the entire interval?

$(-7, -5)$ $\qquad$ $\boxed{(-5, -3)}$ $\qquad$ $(-1, 1)$ $\qquad$ NONE OF THESE

f. [2 points] On which of the following interval(s) is $h''(x)$ decreasing on the entire interval?

$\boxed{(-7, -5)}$ $\qquad$ $(-5, -3)$ $\qquad$ $(-1, 1)$ $\qquad$ NONE OF THESE

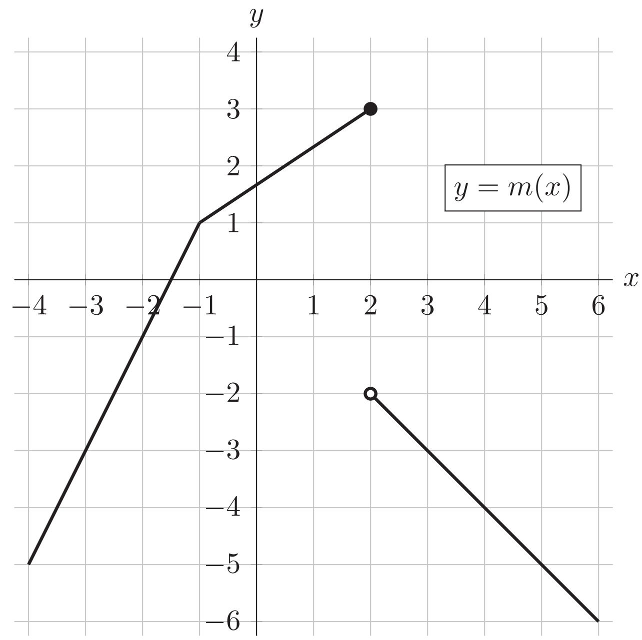
Note: exam problem numbering is off by 1

3. [11 points]

The function $m(x)$ is defined on $(-\infty, \infty)$.

A portion of the graph of the function $m(x)$ is shown.

Note that $m(x)$ is linear on the intervals $(-4,-1)$, $(-1,2)$ and $(2,6)$.



a. [8 points] Evaluate each of the following quantities **exactly**, or write DNE if the value does not exist. You do not need to show work, but limited partial credit may be awarded for work shown. Your answers should not contain the letter m, but do not need to be fully simplified.

i. [2 points] Let $v(x) = x^3 m(x)$. Compute $v'(4)$.

Solution: $v'(x) = 3x^2 m(x) + x^3 m'(x)$. Thus $v'(4) = 48 * (-4) + 64(-1) = -256$.

Answer: $v'(4) =$ -256

ii. [2 points] Let $u(x) = 5m(x-1) + 8$. Compute $u'(3)$.

Solution: $u(x)$ is not continuous at $x = 3$.

Answer: $u'(3) =$ **DNE**

iii. [2 points] Let $w(x) = \dfrac{1}{m(x)}$. Compute $w'(-3)$.

Solution: $w'(x) = -\dfrac{m'(x)}{m(x)^2}$. Thus $w'(-3) = -\frac{2}{9}$.

Answer: $w'(-3) =$ $-\frac{2}{9}$

iv. [2 points] Let $r(x) = \sin(\pi m(x))$. Compute $r'(-2)$.

Solution: $r'(x) = \pi m'(x) \cos(\pi m(x))$. Thus $r'(-2) = 2\pi \cos(-\pi) = -2\pi$.

Answer: $r'(-2) =$ -2π

b. [3 points] Suppose $j(x)$ is a function whose **derivative** is given by the above graph (i.e. $j'(x) = m(x)$). Find a formula for $Q(x)$, the quadratic approximation of $j(x)$ at $x = 5$, assuming $j(5) = 4$.

Answer: $Q(x) =$ $-\frac{1}{2}(x-5)^2 - 5(x-5) + 4$

Note: exam problem numbering is off by 1

4. [12 points]
Suppose $h(x)$ is a continuous function defined for all real numbers x. The **derivative** and **second derivative** of $h(x)$ are given by

$$h'(x) = (x-13)^2(x+4)^{3/7} \qquad \text{and} \qquad h''(x) = \frac{17(x-13)(x+1)}{7(x+4)^{4/7}}.$$

a. [6 points] Find the x-coordinates of all local extrema of $h(x)$. If there are none of a particular type, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

Solution: The critical points of $h(x)$ are at $x = 13, -4$. Applying the first derivative test we have:

	$x < -4$	$-4 < x < 13$	$x > 13$
$h'(x)$	$+\cdot - = -$	$+\cdot + = +$	$+\cdot + = +$

Answer: Local max(es) at x = **None** Local min(s) at x = -4

b. [6 points] Find the x-coordinates of all inflection points of $h(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: The second derivative is zero at $x = 13, -1$ and undefined at $x = -4$. We need to check if the sign of $h''(x)$ changes at these points.

	$x < -4$	$-4 < x < -1$	$-1 < x < 13$	$x > 13$
$h''(x)$	$\frac{-\cdot -}{+} = +$	$\frac{-\cdot -}{+} = +$	$\frac{-\cdot +}{+} = -$	$\frac{+\cdot +}{+} = +$

Answer: Inflection Point(s) at x = $-1, 13$

Note: exam problem numbering is off by 1

5. [12 points] Isabelle is a bee keeper who wants to sell honey at the local farmers market. Let $y = H(d)$ be the amount of honey, in pounds, that Isabelle will sell in a month if she charges d dollars per pound of honey. The functions $H(d)$ and $H'(d)$ are defined and differentiable for all $d \geq 0$. Some values are given in the table below.

d	5.00	5.75	6.50	7.25	8.00	8.75
$H(d)$	59	52	46	38	29	23
$H'(d)$	-10.4	-9.1	-7.8	-11.0	-12.2	-7.6

Assume that $H(d)$ is decreasing and that between each pair of consecutive values of d given in the table, $H'(d)$ is either always increasing or always decreasing.

a. [3 points] Write a formula for the linear approximation $L(d)$ of $H(d)$ near $d = 6.50$, and use it to estimate the amount of honey, in pounds, Isabelle will sell if she charges \$6.30 per pound.

Answer: $L(d) =$ $-7.8(d - 6.5) + 46$

Answer: $\approx$ \$47.56

b. [2 points] Is your estimate from the previous part an overestimate, an underestimate, neither, or is there not enough information to decide? Briefly explain your answer.

Solution: At $d = 6.3$, $H'(d)$ is increasing. Thus $H(d)$ is concave up, and so the linear approximation is an underestimate.

c. [3 points] Write a formula for the linear approximation $K(y)$ of $(H^{-1})(y)$ near $y = 31$.

Solution: Not enough information due to a typo; all students earned these 3 points on the exam.

d. [2 points] Use the table to approximate $H''(8.75)$.

Answer: $H''(8.75) \approx$ $\frac{H'(8.75) - H'(8)}{8.75 - 8} \approx 6.133$

e. [2 points] The hypotheses of the Mean Value Theorem are satisfied for $H(d)$ on the interval $[5.00, 5.75]$. The conclusion of the theorem then tells you that there is a c in the interval $[5, 5.75]$ so that

$H'(c)$ $=$ $\frac{H(5.75) - H(5)}{5.75 - 5} \approx -9.33$

Note: exam problem numbering is off by 1

6. [10 points] Let $P = F(t)$ be the size, in thousands of people, of a certain band's fan club t years after the beginning of 2020. Formulas modeling $F(t)$ and $F'(t)$, the **derivative** of $F(t)$, are given below.

$$F(t) = 175 + 35(t^3 - 7t^2 + 13t - 5)e^{-t} \quad \text{and} \quad F'(t) = -35(t-1)(t-3)(t-6)e^{-t}.$$

In both parts below, you must use calculus to find your answers, and be sure to show enough evidence to fully justify your answers. For each answer blank, write NONE if appropriate.

a. [5 points] During the first two years after the beginning of 2020 (i.e. for $0 \le t \le 2$), when will the band's fan club have the largest and the smallest number of members?

Solution: The only critical point in the domain is $t = 1$. Testing the value of $F(t)$ at the critical point and endpoints, we have

$$F(0) = 0$$
$$F(1) = 175 + \frac{70}{e} \approx 200.75$$
$$F(2) = 175 + \frac{35}{e^2} \approx 179.74$$

Answer: Largest at t = **1** Smallest at t = **0**

b. [5 points] After the beginning of 2022 (i.e. for $t \ge 2$), what are the largest and smallest number of members the band's fan club will have?

Solution: The critical points in the domain are $t = 3$ and $t = 6$. Testing the value of $F(t)$ at the critical points and determining the end behavior, we have

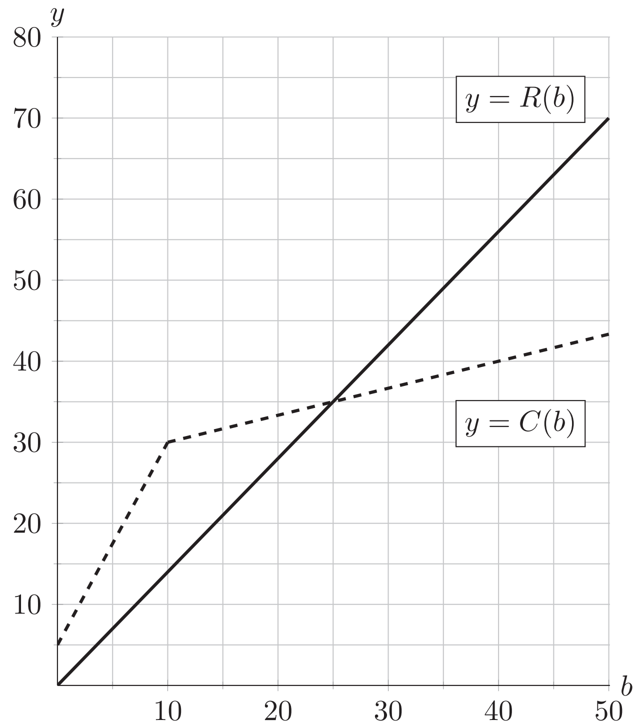
$$F(2) = 175 + \frac{35}{e^2} \approx 179.74$$
$$F(3) = 175 - \frac{70}{e^3} \approx 171.51$$
$$F(6) = 175 + \frac{1295}{e^6} \approx 178.21$$
$$\lim_{t\to\infty} F(t) = 175$$

Answer: Largest number of members, in thousands: **179.74**

Answer: Smallest number of members, in thousands: **171.51**

Note: exam problem numbering is off by 1

7. [10 points] A local bakery makes and sells bagels. When they make and sell b bagels in a given day, their cost is $C(b)$ dollars and their revenue is $R(b)$ dollars. Below, $R(b)$ is graphed as solid line, while $C(b)$ is graphed as a dashed line. Note that the bakery only has the capability to make up to 50 bagels each day.



a. [1 point] What are the company's fixed costs, in dollars?

Answer: 5

b. [2 points] What is the selling price, in dollars, of each bagel?

Answer: 1.4

c. [2 points] Find the marginal cost, in dollars per bagel, at $b = 20$ bagels.

Answer: $\frac{1}{3}$

d. [2 points] Estimate the bakery's daily profit, in dollars, if they produce and sell 40 bagels.

Answer: ≈ 16

e. [2 points] How many bagels should the bakery produce and sell each day if they want to maximize their profit?

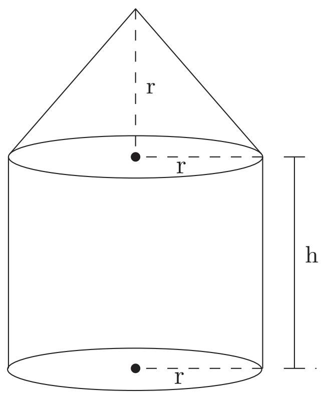
Answer: 50

f. [1 point] How many bagels would the bakery have to produce and sell each day to **minimize** their profit (that is, maximize their losses)?

Answer: 10

Note: exam problem numbering is off by 1

8. [9 points]



A city is in the planning stages of building a shed to store road salt. One design being considered is shown above. The sides would be a cylinder of radius r feet and height h feet, and the roof would be a cone in which both the radius and height are equal to r feet. (The city does not need to build a floor.) The cost of the materials for this shed, in dollars, is

$$4\pi r^2 + 4\pi rh.$$

If the city wants to spend \$20,000 on materials, what values of r and h will maximize the volume of the shed? Give your answers to at least two decimal places, and be sure to find and justify your answers using calculus.
Note that the volume of a cone with radius R and height H is $\frac{1}{3}\pi R^2 H$.

Solution: The total amount, in dollars, the city spends on a shed of height h and radius r is given by

$$20,000 = 4\pi r^2 + 4\pi rh.$$

Solving for h, we obtain

$$h = \frac{20,000 - 4\pi r^2}{4\pi r}.$$

Noting that the height of the cone is the same as its height, we have that the total volume of the shed is given by

$$V = \pi r^2 h + \frac{1}{3}\pi r^3.$$

Substituting our formula for h, we obtain

$$V = \pi r^2 \frac{20,000 - 4\pi r^2}{4\pi r} + \frac{1}{3}\pi r^3.$$

This then simplifies to

$$V = 5000r - \pi r^3 + \frac{1}{3}\pi r^3.$$

Taking the derivative, we obtain

$$V' = 5000 - 3\pi r^2 + \pi r^2 = 5000 - 2\pi r^2.$$

This is defined everywhere, and we find that the derivative vanishes at $\pm\sqrt{2500/\pi}$. We know that r cannot be negative, and when $r = \sqrt{2500/\pi}$, we have $h = \dfrac{10,000}{4\pi\sqrt{2500/\pi}} = \sqrt{2500/\pi}$, both of which are positive, meaning the critical point is in our domain.
To show that volume is maximized, we only need to show that our lone critical point is the location of a local max (since it is the only critical point). To do this, we use the second derivative test. We have

$$V'' = -4\pi r$$

, which is negative at $r = \sqrt{2500/\pi}$, meaning that by the second derivative test we do in fact have a local max.
Thus the volume is maximized when $r = \sqrt{2500/\pi} \approx 28.21$ feet and $h = \dfrac{10,000}{4\pi\sqrt{2500/\pi}} \approx 28.21$.

Note: exam problem numbering is off by 1

9. [5 points]
A curve is implicitly defined by the equation

$$\ln(kx) - 3xy^2 = \pi,$$

where k is a constant. Compute $\frac{dy}{dx}$. Your answer may include k. Show every step of your work.

Solution: Taking the derivative, we have

$$\frac{1}{x} - 3y^2 - 6xyy' = 0.$$

Thus

$$\frac{1}{x} - 3y^2 = 6xyy',$$

and so

$$y' = \frac{\frac{1}{x} - 3y^2}{6xy}.$$

Answer: $\frac{dy}{dx} =$ $\frac{\frac{1}{x} - 3y^2}{6xy}$

Note: exam problem numbering is off by 1

10. [9 points] The function $g(x)$ is given by the equation

$$g(x) = \begin{cases} 3|x+2| - 8x - 11 & x \le -1 \\ x^2 - 3x - 4 & -1 < x < 2 \\ 12(x-10)^{1/3} + 2x + 14 & x \ge 2. \end{cases}$$

You must show work for parts a–d of this problem.

Solution: First note that we have

$$g'(x) = \begin{cases} -11 & x < -2 \\ -5 & -2 < x < -1 \\ 2x - 3 & -1 < x < 2 \\ 4(x-10)^{-2/3} + 2 & x > 2, x \neq 10. \end{cases}$$

(1) Is $g(x)$ continuous at -1?

Solution: Yes.

$$3|-1+2| + 8 - 11 = 0 = (-1)^2 + 3 - 4.$$

(2) Is $g(x)$ differentiable at -1?

Solution: Yes.

$$3 - 8 = -5 = 2(-1) - 3.$$

(1) Is $g(x)$ continuous at 2?

Solution: Yes.

$$(2)^2 - 6 - 4 = -6 = 12(-8)^{1/3} + 4 + 14.$$

(2) Is $g(x)$ differentiable at 2?

Solution: No.

$$2 * 2 - 3 = 1 \neq 3 = 4(-8)^{-2/3} + 2.$$

(3) List **all** points at which $g(x)$ is not differentiable.

Solution: It is not differentiable at $x = -2, 2, 10$. There is a corner at $x = -2$ and a vertical tangent line at $x = 10$.