Math 115 — Final Exam — ONLINE, April 2020

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 9 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	5	
2	13	
3	12	
4	11	
5	12	

Problem	Points	Score
6	14	
7	13	
8	7	
9	13	
Total	100	

1. [5 points] You have **two hours** to complete this exam, which is 5 pages and has 9 problems (including this one) totaling 100 points. You may use a graphing calculator (according to the requirements on the course website), as well as your textbook or ebook and any notes. Electronic notes such as pdf files are allowed if they were downloaded in advance.

This final exam is to be completed **without** use of the internet (except to access this exam and submit work). You may **not** use help from other individuals (other students, tutors, online help forums, etc.), and will **not** communicate with any other person about the exam until 9pm today (i.e., 9pm EDT on Monday, April 27).

As your submission for this problem, you must write "I agree", sign your name, and write your UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

2. [13 points] Suppose that $h(x)$ is invertible and twice differentiable. Some values of $h(x)$ and its derivatives are listed in the table below. Missing values are denoted by a "?".

x	0	2	4	6	8	10
$h(x)$	-1	0.4	?	2	3.4	?
$h'(x)$	1.2	0	0.8	1.2	0.6	?
$h''(x)$	-0.8	0	0	?	-0.2	-1

You do not need to show work for this problem, but limited partial credit may be awarded for work shown. Answer each of the following, or if there is not enough information, write NEI.

a. [2 points] Let $a(x) = h(8e^{3-x})$. Find $a'(3)$.

Solution: $a'(x) = -8e^{3-x}h'(8e^{3-x})$, so $a'(3) = -8h'(8) = -8(0.6) = -4.8$.

b. [2 points] Let $b(x) = h(x)h'(x)$. Find $b'(0)$.

Solution: $b'(x) = h(x)h''(x) + (h'(x))^2$, so $b'(0) = -1(-0.8) + (1.2)^2 = 2.24$.

c. [2 points] Let $c(x) = h^{-1}(x)$. Find $c'(2)$.

Solution: $c'(x) = \dfrac{1}{h'(h^{-1}(x))}$, so $c'(2) = \dfrac{1}{h'(6)} = \dfrac{1}{1.2} = \dfrac{5}{6}$.

d. [2 points] The tangent line to $h(x)$ at $x = 10$ is given by $y = 0.4x + 8$. Find $h(10)$ and $h'(10)$.

Solution: $h(10) = 12$ and $h'(10) = 0.4$

e. [2 points] Find $\displaystyle\lim_{k\to 0} \frac{h(4+k) - h(4)}{k}$.

Solution: $h'(4) = 0.8$

f. [3 points] Assume that between each consecutive pair of columns in the table, the values of $h''(x)$ are either always positive or always negative. Which of the values $x = 0, 2, 4, 6, 8, 10$ must be inflection points of $h(x)$?

Solution: $x = 2$ and $x = 6$

3. [12 points] In parts of Antarctica, snowfall accumulates each year and is eventually compacted into ice. A research team is drilling down into this ice to collect a sample, called an ice core, of snowfall from past years.

- Let $D(t)$ be the depth below the surface, in feet, that the drill has reached t minutes after it begins drilling the ice core.
- Let $A(p)$ be the age, in years, of the ice at a depth of p feet below the surface.

The functions $D(t)$ and $A(p)$ are invertible and differentiable. Use a complete sentence to write a practical interpretation for the equations in **a.–c.**

a. [3 points] $D^{-1}(A^{-1}(110)) = 35$

Solution: The drill reaches ice that is 110 years old 35 minutes after it begins drilling the ice core.

b. [3 points] $A'(185) = 12$

Solution: The ice 186 feet below the surface is approximately 12 years older than the ice 185 feet below the surface.

c. [3 points] $\displaystyle\int_{60}^{120} D'(t)dt = 172$

Solution: Two hours after it begins drilling the ice core, the drill is 172 feet deeper than it was after only one hour of drilling.

d. [3 points] Write an expression involving an integral that represents the average age of the ice in the first 300 feet below the surface.

Solution: $\displaystyle\frac{1}{300}\int_{0}^{300} A(p)dp$

4. [11 points] A stalagmite is a rock formation that rises from the floor of a cave, while a stalactite is a rock formation that hangs from the cave's ceiling. Throughout this problem, be sure your work is clear.

a. [6 points] A certain stalagmite, in the shape of a cone as shown below at left, is growing both in radius and in height. When the volume of the stalagmite is 36π cubic inches, how fast is the volume of the stalagmite growing, in cubic inches per year, if at that time the radius is growing by $\frac{1}{200}$ inches per year, the height is 12 inches, and the height is growing at a rate of $\frac{1}{500}$ inches per year?

Note that the volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$.

Solution: First, note that since $V = \frac{1}{3}\pi r^2 h$, we have that $36\pi = \frac{1}{3}\pi r^2 12$. Thus $r = 3$. Now taking the derivative of the volume equation with respect to time, we have

$$V' = \frac{1}{3}\pi(2rr'h + r^2h').$$

Plugging in our values of r, r', h, and h', we have $V' = \frac{1}{3}\pi(2 * 3 * \frac{1}{200} * 12 + 9 * \frac{1}{500}) = \frac{63\pi}{500} \approx 0.396$ cubic inches per year.

b. [5 points] A certain stalactite, also in the shape of a cone, has a fixed radius of 3 inches, as shown below at right, but its height is growing. How fast is the height of the stalactite growing when the height is 18 inches, if at this time the area of the sides of the stalactite (not including the circular base) is growing at a rate of 0.2 square inches per year? Include units.

Note that the area of the sides (not including the circular base) of a cone of radius r and height h is $\pi r\sqrt{h^2 + r^2}$.

Solution: We have $A = \pi r\sqrt{h^2 + r^2} = 3\pi\sqrt{h^2 + 9}$ since the height is constant. Taking the derivative of both sides with respect to time, we have $A' = \dfrac{3\pi(2hh')}{2\sqrt{h^2 + 9}}$. Using the given values of A' and h, we have

$$0.2 = \frac{6 * 18\pi h'}{2\sqrt{18^2 + 9}} = \frac{108\pi h'}{2\sqrt{333}}.$$

Thus $h' = \dfrac{0.4 * \sqrt{333}}{108\pi} \approx 0.022$ inches per year.

5. [12 points] Consider the continuous function

$$g(x) = \begin{cases} -(x^2 + 12x + 37)e^{-x} + 17 & x \leq 0 \\ 7x^3 - 21x^2 - 168x - 20 & x > 0. \end{cases}$$

Note that

$$g'(x) = \begin{cases} (x+5)^2 e^{-x} & x < 0 \\ 21(x-4)(x+2) & x > 0. \end{cases}$$

a. [3 points] Find the critical points of $g(x)$.

Solution: The derivative is zero at $x = -5$ and 4. The derivative is undefined at $x = 0$, as the derivative approaches 25 to the left of zero but -168 to the right of zero.

b. [4 points] Find the x-coordinate of all local extrema of $g(x)$, and classify each as a local maximum or a local minimum. Use calculus to find and justify your answers, and be sure to show enough evidence that you have found all local extrema.

Solution:
We will use the first derivative test.

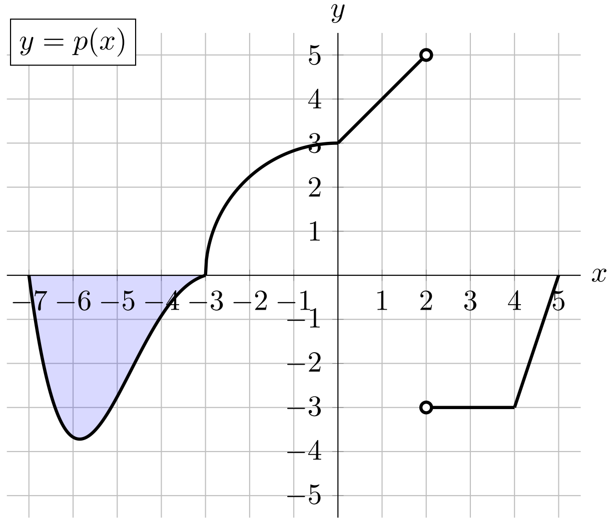
	$x < -5$	$-5 < x < 0$	$0 < x < 4$	$4 < x$
$g'(x)$	$+ \cdot + = +$	$+ \cdot + = +$	$- \cdot + = -$	$+ \cdot + = +$

Thus we have a local max at $x = 0$ and a local min at $x = 4$.

c. [5 points] Find the x-coordinate of all global extrema of $g(x)$ on the interval $(-\infty, 8]$, and classify each as a global maximum or a global minimum. Use calculus to find your answers, and be sure to show enough evidence to fully justify your answers.

Solution: We check the value of $g(x)$ at its critical points as well as the end behavior.
$\lim\limits_{x \to -\infty} g(x) = -\infty$
$g(-5) = -2e^5 + 17 \approx -279.826$
$g(0) = -20$
$g(4) = -580$
$g(8) = 876$
Thus there is no global min, and the global max is at $x = 8$.

6. [14 points] A portion of the graph of a function $p(x)$ is shown below. The area of the shaded region is 8, and the portion of the graph on the interval $[-3, 0]$ is a quarter circle. Also note that $p(x)$ is linear on the intervals $(0, 2)$ and $(4, 5)$.



Let $P(x)$ be the continuous antiderivative of $p(x)$ passing through the point $(0, 1)$.

a. [3 points] Find all critical points of $P(x)$ in the interval $(-7, 5)$. For each, determine if it a local maximum, local minimum, or neither.

Solution: $x = -3$ is a local min, and $x = 2$ is a local max.

b. [2 points] For what values of x in the interval $(-7, 5)$ is $P(x)$ a linear function? Give your answer as one or more intervals.

Solution: $2 < x < 4$.

c. [2 points] For approximately what values of x in the interval $(-7, 5)$ is the function $P(x)$ concave up? Give your answer as one or more intervals.

Solution: $-5.8 < x < 2$ and $4 < x < 5$

d. [2 points] For approximately what values of x in the interval $(-7, 5)$ is the function $p''(x)$ positive? Give your answer as one or more intervals.

Solution: $-7 < x < -4.5$

e. [5 points] Create a table giving the **exact** values of $P(x)$ at $x = -7, -3, 0, 2, 4,$ and 5.

Solution: $P(-7) = 9 - \frac{9}{4}\pi$
$P(-3) = 1 - \frac{9}{4}\pi$
$P(0) = 1$
$P(2) = 9$
$P(4) = 3$
$P(5) = 1.5$

7. [13 points] Suppose that $f(t)$ is a differentiable, increasing function defined for all real numbers. Some values of $f(t)$ are listed in the table below. Assume that $f'(t)$ is continuous.

t	2.5	3.1	4.0	4.5	5.5	7.0	8.5	9.4	10	10.5	12.5
$f(t)$	3.2	4.5	6.5	7.2	8.5	9.2	9.8	10.5	11.2	12.5	14.5

a. [2 points] Compute $\displaystyle\int_4^7 f'(t)dt$ **exactly**, or write NEI if there is not enough information to do so.

Solution: $\displaystyle\int_4^7 f'(t)dt = f(7) - f(4) = 9.2 - 6.5 = 2.7$

b. [2 points] Compute the average value of $f'(t)$ on the interval $[4.5, 10]$ **exactly**, or write NEI if there is not enough information to do so.

Solution: $\displaystyle\frac{1}{10-4.5}\int_{4.5}^{10} f'(t)dt = \frac{f(10)-f(4.5)}{10-4.5} = \frac{11.2-7.2}{5.5} \approx 0.727.$

c. [2 points] Estimate $f'(9)$.

Solution: $\displaystyle\frac{f(9.4)-f(8.5)}{9.4-8.5} = \frac{10.5-9.8}{0.9} = \frac{7}{9} \approx 0.78$

d. [2 points] Use a left-hand Riemann sum with five equal subdivisions to estimate $\displaystyle\int_{2.5}^{10} f(t)dt$. Write out all the terms in your sum.

Solution: $1.5(f(2.5)+f(4)+f(5.5)+f(7)+f(8.5)) = 1.5(3.2+6.5+8.5+9.2+9.8) = 1.5(37.2) = 55.8$

e. [2 points] Does your answer to part **d.** overestimate, underestimate, or equal the value of $\displaystyle\int_{2.5}^{10} f(t)dt$? Explain your answer.

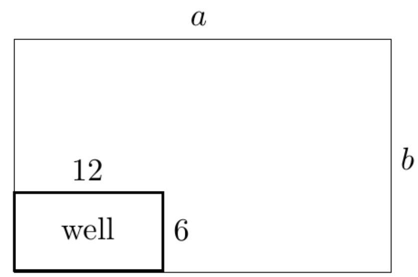
Solution: $f(t)$ is increasing, so the left-hand sum is an underestimate.

f. [3 points] Use a right-hand Riemann sum with four equal subdivisions to estimate $\displaystyle\int_{4.5}^{12.5} f^{-1}(t)dt$. Write out all the terms in your sum.

Solution: $2(f^{-1}(6.5)+f^{-1}(8.5)+f^{-1}(10.5)+f^{-1}(12.5)) = 2(4+5.5+9.4+10.5) = 2(29.4) = 58.8$

8. [7 points] Gretchen wants to build a rectangular garden that includes a well on her property. The well sits on a 12 foot by 6 foot concrete base. Gretchen's plans for the garden, which will have length a feet and width b feet, are as shown.

Gretchen plans to build a rectangular fence around her entire garden, including on the two outside edges of the well. In addition, she wants the usable area of the garden, that is, the area in the garden other than the base of the well, to be 600 square feet.



a. [2 points] Find a formula for a in terms of b.

Solution: The usable area of the garden is given by $600 = ab - 72$. Thus $a = 672/b$

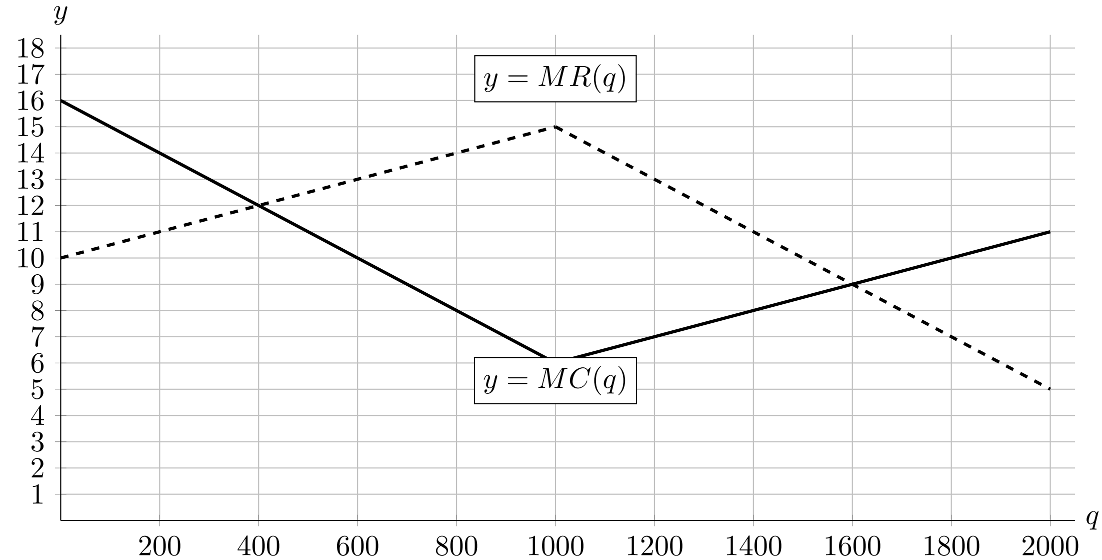
b. [2 points] Find a formula for $F(b)$, the amount of fence, in feet, that Gretchen needs to build her garden. Your formula should be in terms of b only.

Solution: The amount of fence used is equal to the perimeter, or $2a + 2b$. Substituting, we have $F(b) = 2(672/b) + 2b = 1344/b + 2b$.

c. [3 points] What is the domain of the function $F(b)$?

Solution: We need $a > 12$ and $b > 6$. When $a = 12$, our formula from the first part tells us that $b = 672/12 = 56$. Thus the domain is $(6, 56)$.

9. [13 points] The graph below shows the marginal revenue MR (dashed) and marginal cost MC (solid), in dollars per book, for Zelda to print q copies of a certain book. The machinery Zelda needs to start printing costs 800 dollars, but there are no other fixed costs.



You do not need to show work for this problem.

a. [1 point] At what value(s) of q in the interval $[0, 2000]$ is marginal revenue maximized?

Solution: $q = 1000$

b. [2 points] At what value(s) of q in the interval $[0, 2000]$ is cost minimized?

Solution: $q = 0$

c. [2 points] How many books should Zelda print in order to maximize her profit?

Solution: 1600 books

d. [2 points] At which values of q in the interval in the interval $(0, 2000)$ is profit concave up? Give your answer as one or more intervals.

Solution: $MR - MC$ is increasing for $0 < q < 1000$.

e. [3 points] Write an expression involving one or more integrals for Zelda's profit, in dollars, when she prints 1500 copies of her book. Your expression may involve $MR(q)$ and/or $MC(q)$. Do *not* attempt to evaluate the integral.

Solution: $-800 + \int_0^{1500} (MR(q) - MC(q))dq$

f. [3 points] Suppose that Zelda currently plans to print only 200 copies of the book. If she prints 800 copies of the book instead, will this increase or decrease profit? By how much?

Solution: The change is given by $\int_{200}^{800} (MR(q) - MC(q))dq$. This integral is given by computing the signed area between the curves. Noting that this area consists of two triangles, we can compute it exactly as $-300 + 1200 = 900$, so the profit will increase by 900 dollars.
Alternatively, we could count boxes. Depending on how we count, we find between 1 and 2 boxes between $q = 200$ and 400 and between 5 and 7 boxes between 400 and 800. Since each box has area 200, an estimate should give us an increase between 600 and 1200 dollars.