Math 115 — Second Midterm — March 30, 2021

1. This exam has 7 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. You must use the methods learned in this course to solve all problems.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.33333$ is not.
6. You must write your work and answers on **blank, white, physical paper**.
7. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
8. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points	Score
1	3	
2	12	
3	14	
4	10	
5	10	

Problem	Points	Score
6	10	
7	4	
8	11	
9	16	
10	10	
Total	100	

1. [3 points] **There is work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5” by 11” standard sheet of paper, with whatever you want handwritten (not typed) on both sides.
- You are not allowed to use any other resources, including calculators, other notes, or the book.
- You may not use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may not use help from any other individuals (other students, tutors, online help forums, etc.), and may not communicate with any other person about the exam until **10am on Wednesday, March 31** (Ann Arbor time).
- The one exception to the above communication policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write “I agree,” and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated and will not violate this policy.

2. [12 points] A table of values for a differentiable, invertible function $g(x)$ and its derivative $g'(x)$ are shown below.

x	0	1	2	3	4	5
$g(x)$	0	0.5	1	2	5	6
$g'(x)$	1.9	1.5	2.8	2.5	2.6	3

a. [2 points] Use the table provided to give the best possible estimate of $g''(3.5)$.

For parts **b.** and **c.** below, find the **exact** value. Write DNE if the value does not exist, and write NEI if the quantity exists but there is not enough information provided to compute its value. Your answers should not include the letters g or h but you do not need to simplify. Show work.

b. [3 points] Let $f(x) = g^{-1}(3x)$. Find $f'(2)$.

c. [3 points] Let $k(x) = \dfrac{g(x) - 7}{\ln(x)}$. Find $k'(3)$.

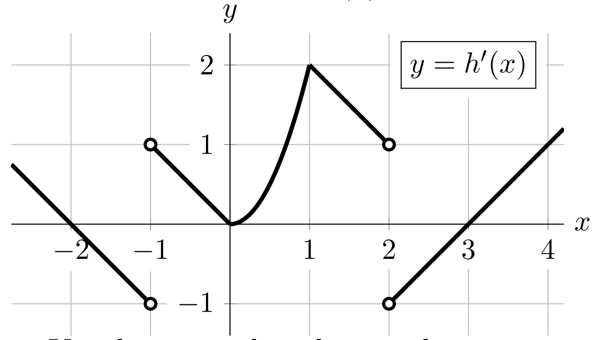
Suppose now that $g(x)$ is the number of thousands of seagulls on a beach when there are x hundred tourists on the beach.

d. [4 points] Complete the following sentence to give a practical interpretation of $(g^{-1})'(1.6) = 0.2$.

If the number of seagulls on the beach increases from 1600 *to* 1605 ...

3. [14 points] A table of values for a differentiable, invertible function $g(x)$ and its derivative $g'(x)$ are shown below to the left. (This is the same table as in the previous problem.) Below to the right is shown a portion of the graph of $h'(x)$, the **derivative** of a function $h(x)$. The function $h(x)$ is defined and continuous for all real numbers.

x	0	1	2	3	4	5
$g(x)$	0	0.5	1	2	5	6
$g'(x)$	1.9	1.5	2.8	2.5	2.6	3



Answer parts **a.–c.**, or write NONE if appropriate. You do not need to show work.

a. [2 points] List the x-coordinates of all critical points of $h(x)$ on the interval $(-2, 4)$.

b. [2 points] List the x-coordinates of all critical points of $h'(x)$ on the interval $(-2, 4)$.

c. [2 points] List the x-coordinates of all local minima of $h(x)$ on the interval $(-2, 4)$.

d. [8 points] A curve is described implictly by the equation

$$yg(x) = e^{h(x)}.$$

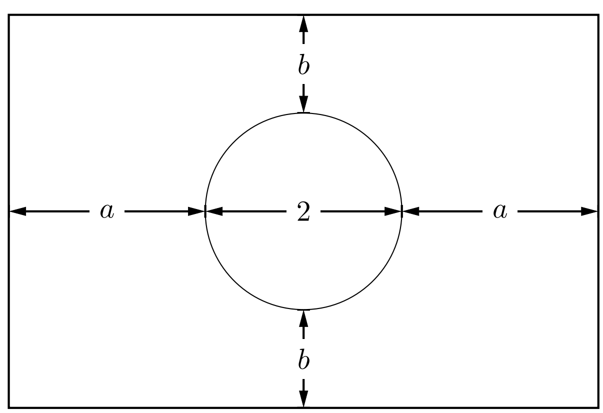
Assume $h(1) = 0$. Then the point $(1, 2)$ lies on this curve.

i. Find $\dfrac{dy}{dx}$ at the point $(1, 2)$. You must show every step of your work.

ii. Write an equation for the tangent line to the curve at the point $(1, 2)$.

4. [10 points] A landscaper is designing a rectangular garden surrounding a circular fountain in the middle.

- The diameter of the fountain is 2 meters.
- The distance from the fountain to the eastern and western edges of the garden is a meters.
- The distance from the fountain to the northern and southern edges of the garden is b meters.
- The part of the garden **outside of the circular fountain** will be covered with exactly 300 square meters of grass.

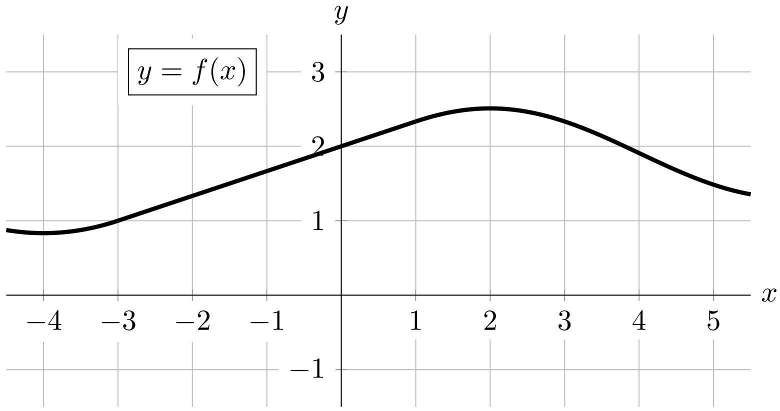


a. [4 points] Write a formula for b in terms of a.

b. [2 points] Write a formula for the function $P(a)$ which gives the rectangular perimeter of the garden in terms of a only.

c. [4 points] In the context of this problem, what is the domain of $P(a)$?

5. [10 points] The graph of the function $f(x)$ is shown below.
Note that $f(x)$ is linear on the interval $(-3, 1)$.

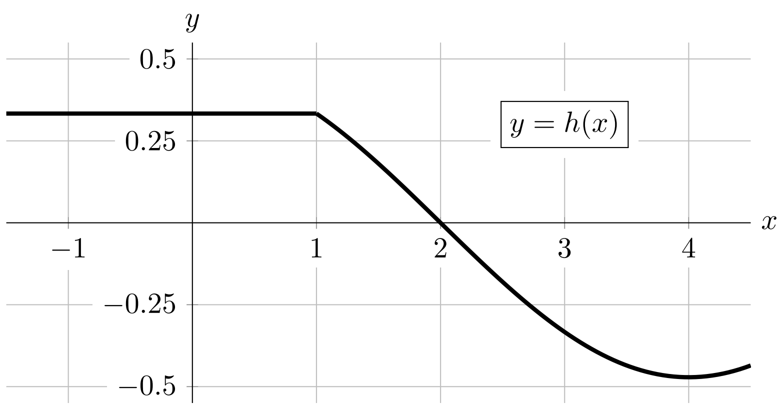


a. [6 points] The function $g(x)$ is given by the equation

$$g(x) = \begin{cases} e^{px} & x \le 0 \\ Cf(x) & x > 0 \end{cases}$$

where C and p are constants and f is as above. Find one pair of **exact** values for C and p such that $g(x)$ is differentiable, or write NONE if there are none. Be sure your work is clear.

Part of the graph of the function $h(x)$ is shown below.



Note that $h(4) = -\dfrac{\sqrt{2}}{3}$.

b. [2 points] Complete the following sentence.

Because the function $h(x)$ satisfies the hypotheses of the mean value theorem on the interval $[2, 4]$, there must be some point c with $2 \le c \le 4$ such that...

c. [2 points] On which of the following intervals does $h(x)$ satisfy the hypotheses of the mean value theorem? List all correct answers, or write NONE.

$[-1, 0]$ $\qquad$ $[0, 3]$ $\qquad$ $[1, 4]$

6. [10 points] A manufacturer is constructing a closed hollow cylindrical tank out of a metal that costs \$2 per square foot. (Note that the tank has both a bottom and a top made of this same metal.) The tank's top must also be coated with a chemical that costs \$5 per square foot. The manufacturer will spend exactly \$180 on the tank.

- Find the height and radius of the cylindrical tank, in feet, so that the tank has the maximum possible volume.
- What is the maximum volume in this case, in cubic feet?

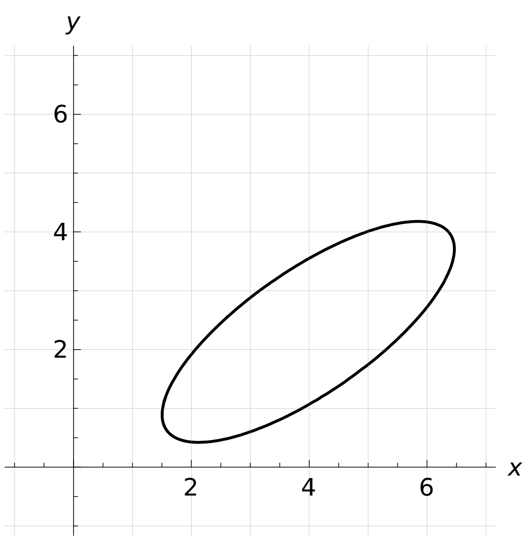
In your solution, make sure to carefully define any variables and functions you use, use calculus to justify your answers, and show enough evidence that the values you find do in fact maximize the volume.

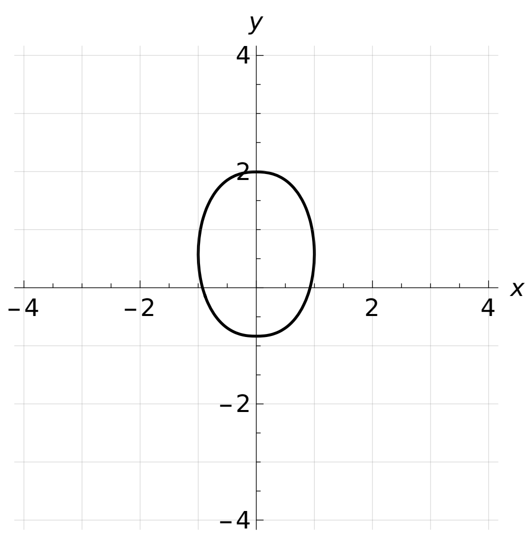
7. [4 points] A curve $\mathcal{C}$ gives y as an implicit function of x and satisfies

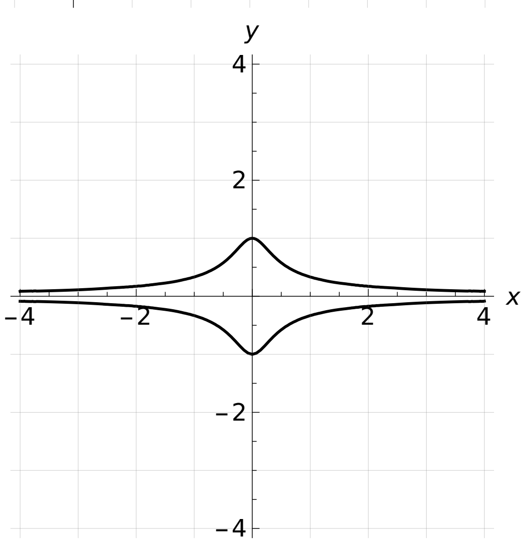
$$\frac{dy}{dx} = \frac{2xy}{3y^2 - x^2} \quad \text{which can be factored and rewritten as} \quad \frac{dy}{dx} = \frac{2xy}{(\sqrt{3}y - x)(\sqrt{3}y + x)}.$$

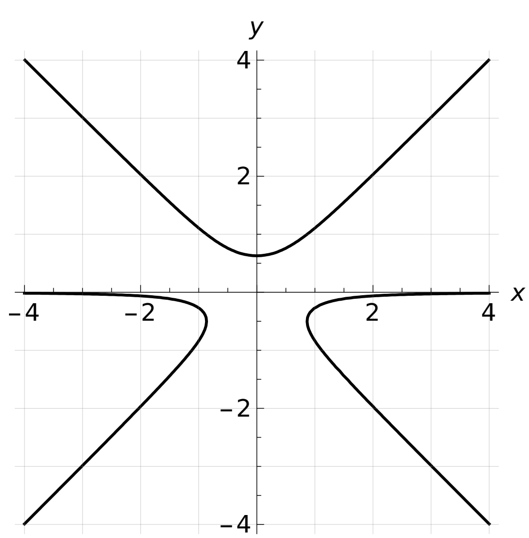
One of the following graphs is the graph of the curve $\mathcal{C}$. Write the letter corresponding to that graph.

Hint: Look for horizontal and vertical tangent lines.

A.


B.


C.


D.


8. [11 points] Suppose $J(x)$ is a continuous function defined for all real numbers x. The **derivative** and **second derivative** of $J(x)$ are given by

$$J'(x) = \frac{x^2(x-1)}{\sqrt[3]{x+4}} \qquad \text{and} \qquad J''(x) = \frac{x(8x^2+31x-24)}{3\left(\sqrt[3]{x+4}\right)^4}.$$

a. [2 points] Find the x-coordinates of all critical points of $J(x)$. If there are none, write NONE.

Throughout parts **b.** and **c.** below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

b. [5 points] Find the x-coordinates of

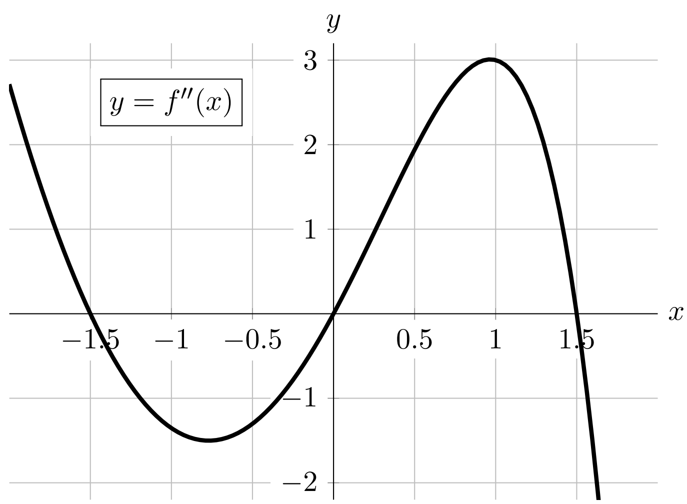
i. all local minima of $J(x)$ and

ii. all local maxima of $J(x)$.

If there are none of a particular type, write NONE.

c. [4 points] The polynomial $8x^2 + 31x - 24$ (from the numerator of $J''(x)$) has two zeroes a and b, where $a \approx -4.54$ and $b \approx 0.66$. How many inflection points does the function $J(x)$ have? Remember to justify your answer. *Hint: What does the graph of* $8x^2 + 31x - 24$ *look like?*

9. [16 points] We consider a function $f(x)$ defined for all real numbers. We suppose that the first and second derivatives $f'(x)$ and $f''(x)$ are also defined for all real numbers. Below we show the graph of the **second derivative** of f. You may assume that $f''(x)$ is decreasing outside of the region shown.



a. [3 points] Find or estimate the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE.

b. [3 points] Find or estimate the x-coordinates of all inflection points of $f'(x)$. If there are none, write NONE.

c. [1 point] Suppose that $f'(0) = 5$. How many critical points does f have?

For parts **d.-f.** below, suppose that $f'(1) = 6.8$ and $f(1) = 4$.

d. [4 points] Let $Q(x)$ be the quadratic approximation of $f(x)$ near $x = 1$. Find a formula for $Q(x)$.

e. [2 points] Is the linear approximation of $f(x)$ near $x = 1$ an overestimate or an underestimate of $f(x)$ for values of x near 1? Explain your reasoning.

f. [3 points] Let $L(x)$ be the linear approximation of $f'(x)$ (the derivative of f) near $x = 1$. Find a formula for $L(x)$.

10. [10 points] Consider a continuous function $f(x)$, and suppose that $f(x)$ and its first derivative $f'(x)$ are differentiable everywhere. Suppose we know the following information about $f(x)$ and its first and second derivatives.

- On the interval $(-\infty, -2)$, we have $f(x) = 2^{-x}$.
- $\lim_{x\to\infty} f(x) = 6$.
- $f(2) = -5$, $f(3) = 7$, and $f(4) = 8$.
- $f'(x)$ is equal to 0 at $x = -1$, 2, 4, and not at any other x-values.
- $f''(x) < 0$ on the intervals $-1 < x < 0$ and $3 < x < 5$, and not on any other interval.

For each part below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [5 points] Find the x-coordinates of

i. the global minimum(s) of $f(x)$ on $[3, \infty)$ and

ii. the global maximum(s) of $f(x)$ on $[3, \infty)$.

If there are none of a particular type, write NONE. If there is not enough information to find a desired x-coordinate, write NEI.

b. [5 points] Find the x-coordinates of

i. the global minimum(s) of $f(x)$ on $(-\infty, \infty)$ and

ii. the global maximum(s) of $f(x)$ on $(-\infty, \infty)$.

If there are none of a particular type, write NONE. If there is not enough information to find a desired x-coordinate, write NEI.