Math 115 — First Midterm — February 23, 2021

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EXAM SOLUTIONS

1. This exam has 14 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. You must use the methods learned in this course to solve all problems.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.33333$ is not.
6. You must write your work and answers on **blank, white, physical paper**.
7. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
8. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points	Score
1	3	
2	12	
3	11	
4	12	
5	10	
6	9	

Problem	Points	Score
7	9	
8	8	
9	7	
10	14	
Total	95	

1. [3 points] **There is work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want handwritten (not typed) on both sides.
- You are not allowed to use any other resources, including calculators, other notes, or the book.
- You may not use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may not use help from any other individuals (other students, tutors, online help forums, etc.), and may not communicate with any other person about the exam until **11pm on Tuesday** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write "I agree," and write your initials and UMID number to signify that you understand and agree to this policy. By doing this and submitting it to Gradescope, you are attesting that you have not violated this policy.

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2. [12 points]

a. [6 points] Consider the given table of values for the function $j(u)$.

u	1	2	3
$j(u)$	6		54

i. Supposing that $j(u)$ is a linear function, fill in the missing entry in the table. Show your work.

Solution: Since $j(u)$ is linear, its output increases by a constant amount m every time we increment u by 1. The table tells us that when we increment u twice, the output increases by $m + m = 54 - 6 = 48$. Therefore $m = 24$, and so $j(2) = j(1) + m = 6 + 24 = 30$.

ii. Supposing that $j(u)$ is an exponential function, fill in the missing entry in the table. Show your work.

Solution: Since $j(u)$ is exponential, its output increases by a constant multiplicative factor a every time we increment u by 1. The table tells us that when we increment u twice, the output increases by a factor of $a \cdot a = 54/6 = 9$. Therefore $a = 3$, and so $j(2) = j(1) \cdot a = 6 \cdot 3 = 18$.

b. [6 points] A radioactive substance decays exponentially in such a way that, if you have some amount of it, then after 15.2 days you will only have a third as much of it remaining. If I have 110 grams of this substance today, how long will I have to wait until I only have 3 grams remaining? *Show every step of your work, and give your final answer in exact form.*

Solution: Let $r(t)$ denote the amount of the substance that I will have, in grams, t days after today. We are told that the substance decays exponentially, so there are constants P_0 and a such that $r(t) = P_0 a^t$. With this setup, P_0 is the amount of the substance, in grams, that I start with, so $P_0 = 110$. We are told that after 15.2 days, I will have a third of my substance remaining, which gives us the equation

$$\begin{aligned} 110/3 &= 110 \cdot a^{15.2} \\ 1/3 &= a^{15.2} \\ (1/3)^{1/15.2} &= a. \end{aligned}$$

To calculate the time t at which I will have 3 grams remaining, we solve the equation

$$\begin{aligned} 3 &= P_0 a^t \\ 3 &= 110 \cdot \left((1/3)^{1/15.2}\right)^t \\ 3/110 &= \left((1/3)^{1/15.2}\right)^t \\ \ln(3/110) &= t \ln\left((1/3)^{1/15.2}\right) \\ \frac{\ln(3/110)}{\ln\left((1/3)^{1/15.2}\right)} &= t \\ \frac{15.2 \ln(3/110)}{\ln(1/3)} &= t. \end{aligned}$$

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3. [11 points] A pilot is flying in an air show. Let $A(t)$ be her altitude, in feet (ft) above the ground, t seconds (sec) after takeoff. Some values of $A(t)$ are shown in the table below, and there is one missing value, denoted by "?".

t	5	22	23	60	60.1	70
$A(t)$	300	1100	1400	400	?	1200

a. [3 points] Use the table to give the best possible estimate of $A'(22)$. Make sure to include the relevant units as part of your answer.

Solution: The best possible estimate of $A'(22)$ is obtained when we calculate the average rate of change over the smallest available interval containing $t = 22$. In this case, the smallest available interval is $[22, 23]$, and so we compute:

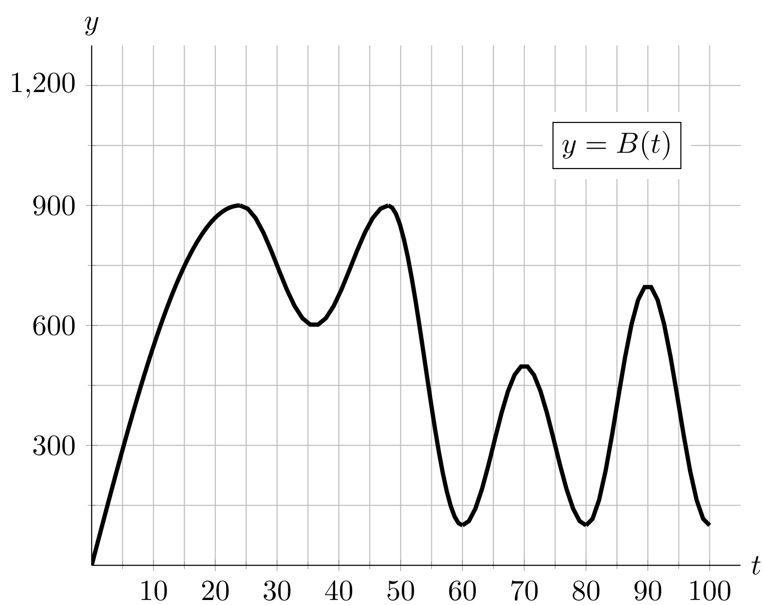
$$\begin{aligned} A'(22) &\approx \text{Average rate of change over } [22, 23] \\ &= \frac{1400 - 1100}{23 - 22} \\ &= \frac{300}{1} \\ &= 300 \text{ feet per second.} \end{aligned}$$

b. [3 points] Suppose that $A'(60) = 550$. Give an approximate value for the missing entry in the table. Make sure to include the relevant units as part of your answer.

Solution: The equation $A'(60) = 550$ means that, when ε is a *small* number, we have $A(60 + \varepsilon) \approx 400 + 550 \cdot \varepsilon$. The missing entry in the table is at $t = 60.1$, so here we may take ε to be the number 0.1.

Then the equation $A'(60) = 550$ tells us that the missing entry $A(60.1)$ in the table is approximately $A(60) + 550 \cdot 0.1 = 400 + 55 = 455$ feet.

c. [5 points] The pilot flies in a different air show a week later. Let $B(t)$ be her altitude, in feet (ft) above the ground, t seconds (sec) after takeoff. A graph of $B(t)$ is shown below.
(Reduced scale for solutions)



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Let the quantities I-V be defined as follows:

I. The number 0.

II. The pilot's average velocity, in ft/sec, between $t = 15$ and $t = 50$.

III. The pilot's instantaneous velocity, in ft/sec, at $t = 55$.

IV. The pilot's average velocity, in ft/sec, between $t = 50$ and $t = 90$.

V. The pilot's instantaneous velocity, in ft/sec, at $t = 85$.

List the quantities I-V in increasing order.

Solution: Since $B(15) < B(50)$ and $B'(85) > 0$, we see that II and V are greater than I. Since $B(50) > B(90)$ and $B'(55) < 0$, we see that III and IV are less than I. Therefore our ordering is

$$(\text{III or IV}) < (\text{III or IV}) < \text{I} < (\text{II or V}) < (\text{II or V}).$$

Glancing at the graph, it appears that II is a shallow positive slope, while V is a steep positive slope. It also appears that IV is a shallow negative slope, while III is a steep negative slope. This suggests the answer

$$\text{III} < \text{IV} < \text{I} < \text{II} < \text{V}.$$

For the purpose of these solutions, we will verify this answer more carefully, just to be sure.

We now decide whether II or V is greater. Observe that $B(15)$ is about 750, and $B(50)$ is a little less than 900. Therefore the pilot's average velocity between $t = 15$ and $t = 50$ (option II) is no more than $\frac{900-750}{50-15} = \frac{150}{35} < \frac{150}{30} = 5$ ft/sec. Observe that $B'(85)$ (option V) appears very large, almost certainly greater than 5. Indeed, we see that it must be larger than the pilot's average velocity between $t = 80$ and $t = 90$. Since $B(80)$ is less than 150 and $B(90)$ is greater than 600, this average velocity is greater than $\frac{600-150}{90-80} = \frac{450}{10} = 45$ ft/sec. Since $45 > 5$, we conclude that V is greater than II.

We now decide whether III or IV is greater. Observe that $B(50)$ is a little less than 900 and $B(90)$ is more than 600. Therefore the pilot's average velocity between $t = 50$ and $t = 90$ (option IV) is greater than $\frac{600-900}{90-50} = \frac{-300}{40} = \frac{-15}{2} > -8$ ft/sec. Observe that $B'(55)$ (option III) appears likely to be much less than -8. Indeed, we see that it must be less than the pilot's average velocity between $t = 50$ and $t = 60$. Since $B(50)$ is greater than 750 and $B(60)$ is less than 150, this average velocity is less than $\frac{150-750}{60-50} = \frac{-600}{10} = -60$ ft/sec. Since $-60 < -8$, we conclude that III is less than IV.

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4. [12 points] Parts **a.** and **b.** below are unrelated.

a. [6 points] Suppose that the temperature in Staunton, Virginia, in degrees Fahrenheit (°F), can be modeled by a sinusoidal function $S(t)$ where t is the time in months since January 1. Note that, for example, August 1 is seven months after January 1. A formula for $S(t)$ is

$$55 - 21\cos\left(\frac{\pi}{6}t\right),$$

i. Using this model, what is the coldest temperature in Staunton?

Solution: The midline of this sinusoidal function is at an output value of 55, and the amplitude is 21. Therefore the lowest value is $55 - 21 = 34$°F.

ii. Using this model, what is the average temperature over the entire year?

Solution: The midline of this sinusoidal function gives us the average temperature, and so the average temperature is 55°F.

iii. At what time t does the temperature first reach "room temperature" (68°F)? *Give your final answer in exact form.*

Solution: To find when the temperature first reaches room temperature, we must solve for t in the following equation:

$$55 - 21\cos\left(\frac{\pi}{6}t\right) = 68.$$

Since we are looking for the *first* time the temperature reaches room temperature (and there is no horizontal shift from cosine in the given function), the arccosine function will give us our desired t-value. We therefore solve:

$$\begin{aligned}
55 - 21\cos\left(\frac{\pi}{6}t\right) &= 68 \\
-21\cos\left(\frac{\pi}{6}t\right) &= 68 - 55 \\
\cos\left(\frac{\pi}{6}t\right) &= -13/21 \\
\text{so one solution is given by } \frac{\pi}{6}t &= \arccos(-13/21) \\
t &= \frac{6}{\pi}\arccos(-13/21).
\end{aligned}$$

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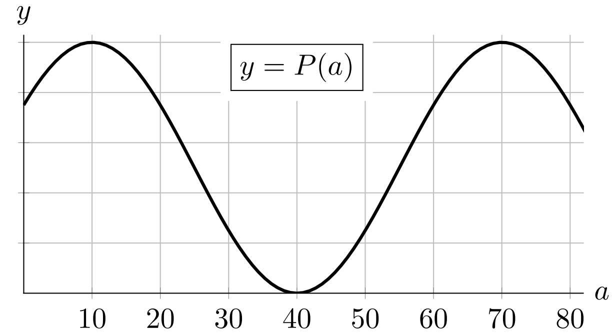
b. [6 points] Suppose that a probe lands on some planet other than Earth, and that its recorded temperature, in degrees Fahrenheit, can be modeled by a sinusdoidal function $P(a)$ where a is the time in years since the probe landed. Note that the scale on the y-axis is unknown.



When the temperature is too cold, the probe is in a state of hibernation. The first time it enters hibernation is at $a = 27$.

i. At what time a does the probe leave hibernation?

Solution: The probe will leave hibernation the *next* time after $a = 27$ that $P(a)$ is equal to $P(27)$. Since P has a minimum at $a = 40$, its graph is symmetric about the vertical line $a = 40$, and so the probe will leave hibernation $40 - 27 = 13$ years after $a = 40$. That is to say, the probe will leave hibernation at $a = 53$.

ii. What is the period of $P(a)$?

Solution: We observe that the function P has a maximum at $a = 10$ and another maximum at $a = 70$. Therefore the period of $P(a)$ is $70 - 10 = 60$.

iii. Use the period you found to calculate the next time at which the probe will enter hibernation.

Solution: The probe will next enter hibernation after one period of $P(a)$ has passed since the first time it entered hibernation. Therefore, the probe will next enter hibernation at $a = 27 + 60 = 87$.

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5. [10 points] Let us consider the following functions, which concern the productivity of a soybean farm. Bushels are a unit of volume often used to measure a farm's yield.

- Let $Y(b)$ be the yield, in bushels of soybeans, of the farm in the year 2019 when it is infested with b beetles.
- Let $R(s)$ be the revenue, in dollars, of the farm in the year 2019 when it yields s bushels of soybeans.

The functions $Y(b)$ and $R(s)$ are differentiable and invertible.

a. [2 points] Use a complete sentence to give a practical interpretation of the equation

$$R(Y(1{,}200)) = 75{,}000.$$

Solution: This equation tells us that when the farm is infested with 1,200 beetles in 2019, its revenue that year is \$75,000.

b. [4 points] Write a single equation representing the following statement in terms of the functions Y, R, and/or their inverses:

> *If there are* 1,600 *beetles, then the farm yields* 200 *bushels of soybeans fewer than are necessary for a revenue of* \$64,000 in the year 2019.

Solution:

$$Y(1{,}600) = R^{-1}(64{,}000) - 200.$$

c. [4 points] Complete the following sentence to give a practical interpretation of the equation

$$Y'(1{,}000) = -0.1.$$

> *If the beetle population was* 1,000 *rather than* 950...

Solution: Since $1{,}000 - 950 = 50$, the equation $Y'(1{,}000) = -0.1$ tells us that $Y(1{,}000)$ is about $0.1 \cdot 50 = 5$ *less* than $Y(950)$. Therefore we may complete the sentence as follows:

> *If the beetle population was* 1,000 *rather than* 950*, the farm would yield about* 5 *fewer bushels of soybeans in* 2019*.*

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6. [9 points] A metal bar is unevenly heated, and a laser thermometer is used to measure its temperature at various points. Let $T(q)$ be the temperature of the bar, in degrees Celsius, q feet from its leftmost end. Some values of $T(q)$ are shown in the table below.

q	1	2	3	4	5	6	7	8	9
$T(q)$	40	70	90	80	60	90	130	100	60

a. [3 points] For which of the following intervals of q-values might the function $T'(q)$ be positive for the entire interval? Give your answer as a list of one or more intervals, or write NONE.

$\boxed{(1,3)}$ $(4,6)$ $\boxed{(5,7)}$ $(7,9)$

Solution: The output $T(q)$ increases from $q = 1$ to $q = 2$, and from $q = 2$ to $q = 3$, and so it *is* possible for the derivative $T'(q)$ to always be positive on $(1, 3)$.

The output $T(q)$ decreases from $q = 4$ to $q = 5$, and so it is *not* possible for the derivative $T'(q)$ to always be positive on $(4, 6)$.

The output $T(q)$ increases from $q = 5$ to $q = 6$, and from $q = 6$ to $q = 7$, and so it *is* possible for the derivative $T'(q)$ to always be positive on $(5, 7)$.

The output $T(q)$ decreases from $q = 7$ to $q = 8$, and from $q = 8$ to $q = 9$, and so it is *not* possible for the derivative $T'(q)$ to always be positive on $(7, 9)$.

b. [3 points] For which of the following intervals of q-values might the function $T(q)$ be concave up for the entire interval? Give your answer as a list of one or more intervals, or write NONE.

$(1,3)$ $\boxed{(4,6)}$ $\boxed{(5,7)}$ $(7,9)$

Solution: From $q = 1$ to $q = 2$, the function $T(q)$ increases by 30, but from $q = 2$ to $q = 3$, it only increases by 20. If $T(q)$ were concave up on $(1, 3)$, this second increase would have been greater than 30. Therefore $T(q)$ is *not* concave up on $(1, 3)$.

From $q = 4$ to $q = 5$, the function $T(q)$ decreases, and from $q = 5$ to $q = 6$, the function $T(q)$ increases. A concave up function has an increasing first derivative, and these two observations do not preclude an increasing first derivative. Therefore $T(q)$ *might* be concave up on $(4, 6)$.

From $q = 5$ to $q = 6$, the function $T(q)$ increases by 30, and from $q = 6$ to $q = 7$, it increases by 40. A concave up function has an increasing first derivative, and these two observations do not preclude an increasing first derivative. Therefore $T(q)$ *might* be concave up on $(5, 7)$.

From $q = 7$ to $q = 8$, the function $T(q)$ decreases by 30, but from $q = 8$ to $q = 9$, the function $T(q)$ decreases by 40. If $T(q)$ were concave up on $(7, 9)$, this second decrease would have been less than 30. Therefore $T(q)$ is *not* concave up on $(7, 9)$.

c. [3 points] What is the average rate of change of $T(q)$ on the interval $2 \le q \le 7$? Include units in your answer.

Solution: The average rate of change of $T(q)$ on the interval $2 \le q \le 7$ is

$$\frac{T(7) - T(2)}{7 - 2} = \frac{130 - 70}{5} = \frac{60}{5} = 12^\circ\mathrm{C/ft}.$$

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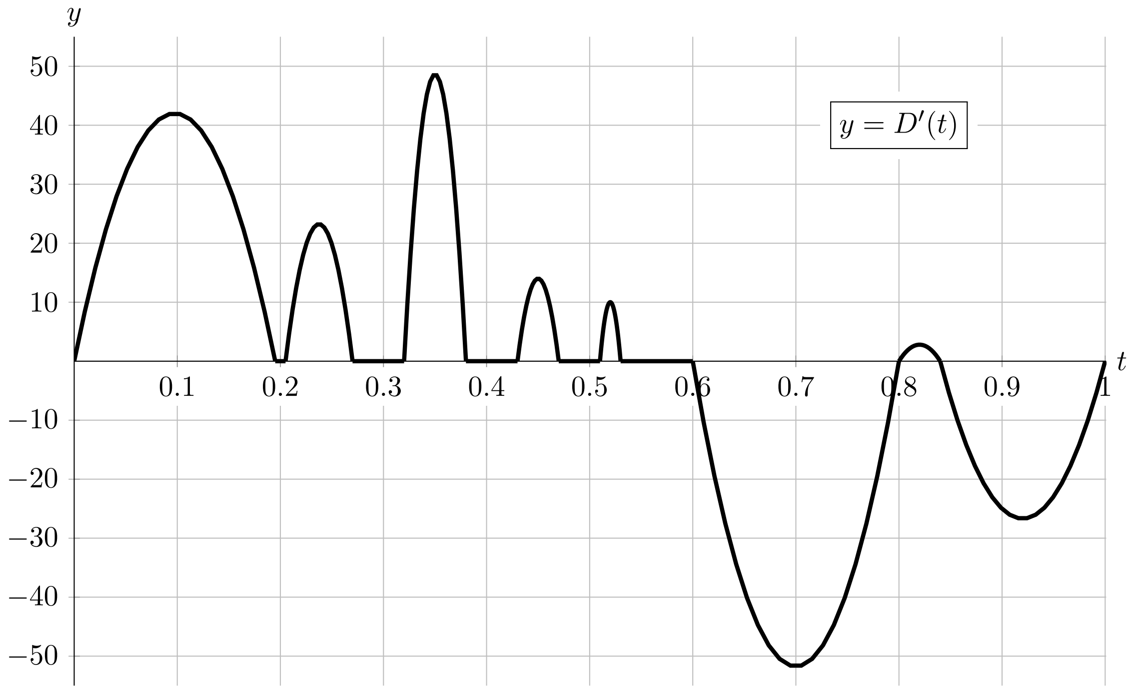
7. [9 points] A pizza delivery driver works for a pizzeria on Main Street, which is a long, straight road. The driver tracks her location with her phone while driving a route on Main Street. Let $D(t)$ be her distance from her pizzeria, in miles, at time t hours after noon. Below is a portion of the graph of $D'(t)$, the **derivative** of $D(t)$.



a. [2 points] On which of the following intervals of t is the driver getting closer to her pizzeria for the entire interval? Give your answer as a list of one or more intervals, or write NONE.

$(0.1, 0.2)$ $(0.2, 0.3)$ $\boxed{(0.6, 0.8)}$ $(0.8, 1)$

Solution: The driver is getting closer to her pizzeria only when $D'(t)$ is *negative* (i.e. her distance to the pizzeria is *decreasing*). The only one of these intervals on which $D'(t)$ is always negative is $(0.6, 0.8)$.

b. [3 points] The speed limit in the driver's hometown is 40 miles per hour. How many different times does she *begin* to drive over the speed limit?

Solution: She *begins* to drive over the speed limit at the t-values where $|D'(t)|$ goes from being smaller than 40 to being larger than 40. The t-values at which this happens are:

- right before $t = 0.1$,
- between $t = 0.3$ and $t = 0.4$,
- right before $t = 0.7$.

Therefore she begins to drive over the speed limit at 3 different times.

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c. [2 points] At which of the following times is the driver farthest from her pizzeria? Write the one best answer.

$t = 0.1$ $t = 0.35$ $t = 0.5$ $\boxed{t = 0.6}$ $t = 0.7$

Solution: Notice that for $0.1 \leq t \leq 0.7$, the function $D'(t)$ starts out having only non-negative values, and then switches to having only non-positive values. Consider some time t. As long as there are more positive values of $D'(t)$ yet to come, the driver will get farther from the pizzeria than she is now! Of course, once the values of $D'(t)$ are negative, she's now going towards the pizzeria, so her furthest point was in the past. The last point where $D'(t)$ has only ever been non-negative is $t = 0.6$, and so this is the time at which she is farthest from the pizzeria.

d. [2 points] Write the number of the the sentence below that best describes the driver's behavior on the interval $0.2 \leq t \leq 0.5$.

1. *The driver keeps returning to the pizzeria to pick up more pizza.*
2. *The driver is driving on a highway without any traffic.*
3. *The driver stops at a series of red lights.*
4. *The driver is driving in circles, looking for a place to park.*

Solution: Let us consider what a graph for each of these options might look like.

1. For this option, we would expect to see $D'(t)$ going from positive to negative over and over again, since she would be leaving ($D'(t) > 0$) and then returning to ($D'(t) < 0$) the pizzeria multiple times.

2. For this option, we would expect $D'(t)$ to never be 0, since she wouldn't stop on the highway unless there is traffic.

3. For this option, we would expect $D'(t)$ to keep switching from positive (driving forward) to 0 (stopped at a light).

4. For this option, we would expect to see $D'(t)$ going from positive to negative over and over again, since she would be driving both towards and away from the pizzeria as she circles around.

Of these descriptions, only number 3 matches the graph, and so this is our answer.

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8. [8 points]

a. [3 points] Let

$$L(q) = \frac{\sin(q)}{1+3q}.$$

Suppose k is a nonzero constant. Write an explicit expression for the average rate of change of L between $q = 5$ and $q = 5 + k$.

Your answer should not involve the letter L. Do not attempt to simplify your expression.

Draw a box around your final answer.

Solution:

$$\frac{\dfrac{\sin(5+k)}{1+3(5+k)} - \dfrac{\sin(5)}{1+3\cdot 5}}{k}$$

b. [5 points] Let

$$P(b) = (\ln(b))^{\tan(b)}.$$

Use the limit definition of the derivative to write an explicit expression for $P'(3)$.

Your answer should not involve the letter P. Do not attempt to evaluate or simplify the limit.

Draw a box around your final answer.

Solution:

$$\lim_{h\to 0} \frac{(\ln(3+h))^{\tan(3+h)} - (\ln(3))^{\tan(3)}}{h}$$

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9. [7 points]
We consider formulas for four different rational functions. Let the formulas I-IV be defined as follows:

I. $\dfrac{x^5(x+1)(x-2)}{x+2}$

II. $\dfrac{x^4(x+1)(x-2)(x-8)}{(x-9)(x-3)}$

III. $\dfrac{(x-9)(x+1)}{x-8}$

IV. $\dfrac{(x-3)(x+3)}{x-9}$

We describe three functions. Match each function below with the formula I-IV that could possibly be the formula for that function. Each function below matches with *exactly one* of the formulas above.

A. The function $g(x)$ is such that $\dfrac{g(x)}{x^5}$ diverges to ∞ as $x \to \infty$.

B. The function $h(x)$ is such that the function

$$S(t) = \begin{cases} \sin(2\pi x) & x < 3 \\ h(x) & x \geq 3 \end{cases}$$

is continuous at $x = 3$.

C. The function $f(x)$ is such that $f(x+3)$ has a vertical asymptote at $x = 5$.

Solution: Description A tells us that the numerator of $g(x)$ must have degree at least 6 greater than the degree of its denominator. In Formula I, the difference in degrees between numerator and denominator is $7 - 1 = 6$, and so already we see that A matches with I. To double check, we see that the difference in degrees for Formula II is $7 - 2 = 5$, for III is $2 - 1 = 1$, and for IV is $2 - 1 = 1$. Therefore Formula I really is the only possible match for Description A.

Description B tells us that $h(3)$ must be equal to $\lim_{x\to 3^-} \sin(2\pi x)$. But $\sin(2\pi x)$ is always continuous, so this limit is equal to $\sin(2\pi \cdot 3) = 0$. We therefore want a formula that evaluates to 0 at $x = 3$. The numerators of our rational functions are all factored into linear factors, and so this means we must choose the rational function with $(x - 3)$ in its numerator. The only option is Formula IV.

Description C tells us that, when we horizontally shift $f(x)$ to the left by 3, there is a vertical asymptote at $x = 5$ for the shifted version of our function. Therefore $f(x)$ itself must have a vertical asymptote at $x = 8$. The denominators of our rational functions are all factored into linear factors, and so this means we must choose the rational function with $(x - 8)$ in its denominator. The only option is Formula III.

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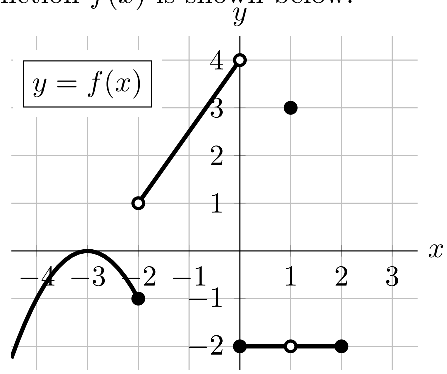
10. [14 points] The graph of the function $f(x)$ is shown below.
(Reduced scale for solutions)



For **a.–b.**, give your answers as a list of one or more of the given numbers, or write NONE

a. [2 points] For which of the values $c = -3, -2, -1, 0, 1$ is $f(x)$ continuous at $x = c$?

Solution: Among these options, $f(x)$ is only continuous when $x = -3$ and when $x = -1$.

b. [2 points] For which of the values $c = -3, -2, -1, 0, 1$ is $\lim_{x\to c^-} f(x) = f(c)$?

Solution: This condition is certainly satisfied at the points where $f(x)$ is continuous, so we already know that $x = -3$ and $x = -1$ should be in our answer. Checking the other points: $\lim_{x\to -2^-} f(x) = -1$ and $f(-2) = -1$. $\lim_{x\to 0^-} f(x) = 4$ but $f(0) = -2$. $\lim_{x\to 1^-} f(x) = -2$ but $f(1) = 3$. So, our final answer is that the desired condition is satisfied at $\boxed{x = -3,\ -2,\ \text{and}\ -1}$.

For **c.–g.**, use the graph of the function $f(x)$ to evaluate each of the expressions below. If a limit diverges to ∞ or $-\infty$ or if the limit does not exist for any other reason, write "DNE." If there is not enough information to evaluate the expression, write "Not enough information."

c. [2 points] $\lim_{x\to 0} f(x)$

Solution: Observe that $\lim_{x\to 0^-} f(x) = 4$ and $\lim_{x\to 0^+} f(x) = -2$. Since the left- and right-hand limits are different, we see that $\lim_{x\to 0} f(x)$ does not exist, so we write "DNE."

d. [2 points] $\lim_{x\to 1} f(x)$

Solution: On either side of $x = 1$, the function $f(x)$ is constant and equal to -2. Therefore $\lim_{x\to 1} f(x) = \boxed{-2}$.

e. [2 points] $\lim_{h\to 0} \dfrac{f(-1+h) - f(-1)}{h}$

Solution: We recognize this expression as $f'(-1)$. Since the function $f(x)$ is linear on the interval $(-2, 0)$, we see that $f'(-1)$ is the slope of this line. We calculate the slope to be $(4-1)/(0-(-2)) = 3/2$.

f. [2 points] $\lim_{x\to 3^+} 4f(x-5) - 1$

Solution: As x approaches 3 from the right, $x - 5$ approaches 2 from the right. This means that $f(x-5)$ approaches 1 from above, and therefore $4f(x-5) - 1$ approaches $4(1) - 1 = \boxed{3}$.

g. [2 points] $\lim_{x\to -3} f(f(x))$

Solution: As x approaches -3 from either side, the value $y = f(x)$ approaches 0 *from negative values only*. Therefore $\lim_{x\to -3} f(f(x)) = \lim_{y\to 0^-} f(y) = \boxed{4}$.

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