Math 115 — Second Midterm — March 30, 2021

EXAM SOLUTIONS

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1. This exam has 16 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. You must use the methods learned in this course to solve all problems.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.33333$ is not.
6. You must write your work and answers on **blank, white, physical paper**.
7. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
8. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points	Score
1	3	
2	12	
3	14	
4	10	
5	10	

Problem	Points	Score
6	10	
7	4	
8	11	
9	16	
10	10	
Total	100	

1. [3 points] **There is work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want handwritten (not typed) on both sides.
- You are not allowed to use any other resources, including calculators, other notes, or the book.
- You may not use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may not use help from any other individuals (other students, tutors, online help forums, etc.), and may not communicate with any other person about the exam until **10am on Wednesday, March 31** (Ann Arbor time).
- The one exception to the above communication policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write "I agree," and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated and will not violate this policy.

2. [12 points] A table of values for a differentiable, invertible function $g(x)$ and its derivative $g'(x)$ are shown below.

x	0	1	2	3	4	5
$g(x)$	0	0.5	1	2	5	6
$g'(x)$	1.9	1.5	2.8	2.5	2.6	3

a. [2 points] Use the table provided to give the best possible estimate of $g''(3.5)$.

Solution: Since $g''(x)$ is the first derivative of $g'(x)$, we may find the best possible estimate of $g''(3.5)$ by calculating the average rate of change of $g'(x)$ over the smallest available interval that contains 3.5. This interval is $[3,4]$, and so we compute

$$g''(3.5) \approx \frac{g'(4)-g'(3)}{4-3} = \frac{2.6-2.5}{1} = \boxed{0.1.}$$

For parts **b.** and **c.** below, find the **exact** value. Write DNE if the value does not exist, and write NEI if the quantity exists but there is not enough information provided to compute its value. Your answers should not include the letters g or h but you do not need to simplify. Show work.

b. [3 points] Let $f(x) = g^{-1}(3x)$. Find $f'(2)$.

Solution: Recall the equation $\frac{d}{dt}\left(g^{-1}(t)\right) = \frac{1}{g'(g^{-1}(t))}$. We use this equation along with the Chain Rule to calculate

$$f'(x) = \frac{1}{g'(g^{-1}(3x))} \cdot 3.$$

Plugging in $x = 2$ gives us

$$f'(2) = \frac{3}{g'(g^{-1}(6))} = \frac{3}{g'(5)} = \frac{3}{3} = \boxed{1.}$$

c. [3 points] Let $k(x) = \frac{g(x) - 7}{\ln(x)}$. Find $k'(3)$.

Solution: Recall the equation $\frac{d}{dx}\ln(x) = \frac{1}{x}$. We use this equation along with the Quotient Rule to calculate

$$k'(x) = \frac{\ln(x)g'(x) - (g(x) - 7) \cdot \frac{1}{x}}{(\ln(x))^2}.$$

Plugging in $x = 3$ gives us

$$\begin{aligned} k'(3) &= \frac{\ln(3)g'(3) - (g(3) - 7) \cdot \frac{1}{3}}{(\ln(3))^2} \\ &= \frac{\ln(3) \cdot 2.5 - (2 - 7) \cdot \frac{1}{3}}{(\ln(3))^2} \\ &= \boxed{\frac{\ln(3) \cdot 2.5 - \frac{5}{3}}{(\ln(3))^2}.} \end{aligned}$$

Suppose now that $g(x)$ is the number of thousands of seagulls on a beach when there are x hundred tourists on the beach.

d. [4 points] Complete the following sentence to give a practical interpretation of $(g^{-1})'(1.6) = 0.2$.

If the number of seagulls on the beach increases from 1600 *to* 1605 ...

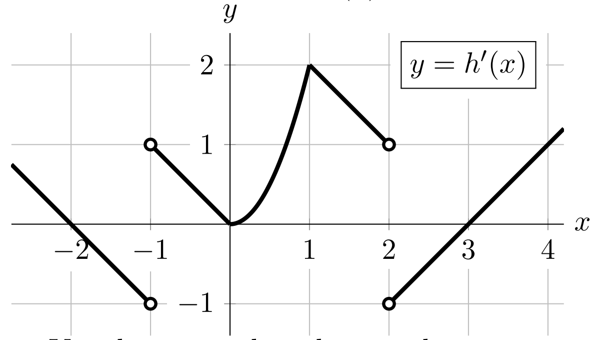
Solution: The function $g^{-1}(s)$ tells us the number of hundereds of tourists on the beach when there are s thousand seagulls. The equation $(g^{-1})'(1.6) = 0.2$ tells us that when there are 1600 seagulls on the beach, then the instantaneous rate of change in the number of tourists is 20 tourists per 1000 seagulls. That is to say, 0.02 tourists per seagull. We therefore may complete the sentence:

If the number of seagulls on the beach increases from 1600 *to* 1605 *the number of tourists increases by about one tenth of a tourist.*

Note: It would also be reasonable to assert that the number of tourists would not change since the change cannot actually be one tenth of a tourist.

3. [14 points] A table of values for a differentiable, invertible function $g(x)$ and its derivative $g'(x)$ are shown below to the left. (This is the same table as in the previous problem.) Below to the right is shown a portion of the graph of $h'(x)$, the **derivative** of a function $h(x)$. The function $h(x)$ is defined and continuous for all real numbers.

x	0	1	2	3	4	5
$g(x)$	0	0.5	1	2	5	6
$g'(x)$	1.9	1.5	2.8	2.5	2.6	3



Answer parts **a.–c.**, or write NONE if appropriate. You do not need to show work.

a. [2 points] List the x-coordinates of all critical points of $h(x)$ on the interval $(-2, 4)$.

Solution: The critical points of $h(x)$ occur where $h'(x)$ either does not exist or is equal to 0. By looking at the graph, we see that this happens when $\boxed{x = -1, 0, 2, \text{ and } 3.}$

b. [2 points] List the x-coordinates of all critical points of $h'(x)$ on the interval $(-2, 4)$.

Solution: Recall that critical points of $h'(x)$ have to be points *in the domain* of $h'(x)$ where the derivative of $h'(x)$ either is 0 or does not exist. We see no places where the derivative of $h'(x)$ is 0, but we do see that the derivative of $h'(x)$ does not exist when $\boxed{x = 0 \text{ and } x = 1.}$

c. [2 points] List the x-coordinates of all local minima of $h(x)$ on the interval $(-2, 4)$.

Solution: We use the first derivative test, and see that $h'(x)$ is negative to the left of $x = -1$, and is positive to the right of $x = -1$. It has this same sort of behavior for $x = 3$. We conclude that $h'(x)$ has local minima at $\boxed{x = -1 \text{ and } x = 3.}$

d. [8 points] A curve is described implictly by the equation

$$yg(x) = e^{h(x)}.$$

Assume $h(1) = 0$. Then the point $(1, 2)$ lies on this curve.

i. Find $\frac{dy}{dx}$ at the point $(1, 2)$. You must show every step of your work.

Solution: We first find an expression for $\frac{dy}{dx}$. We differentiate the left-hand side of our equation using the Product Rule, and we differentiate the right-hand side using the Chain Rule:

$$\frac{d}{dx}\left(yg(x)\right) = \frac{d}{dx}\left(e^{h(x)}\right)$$
$$yg'(x) + \frac{dy}{dx}g(x) = e^{h(x)}h'(x).$$

We plug in $x = 1$ and $y = 2$ to obtain

$$2g'(1) + \frac{dy}{dx}g(1) = e^{h(1)}h'(1)$$
$$2 \cdot 1.5 + \frac{dy}{dx} \cdot 0.5 = e^0 \cdot 2$$
$$3 + \frac{1}{2}\frac{dy}{dx} = 2$$
$$\frac{dy}{dx} = (2 - 3) \cdot 2 = \boxed{-2}.$$

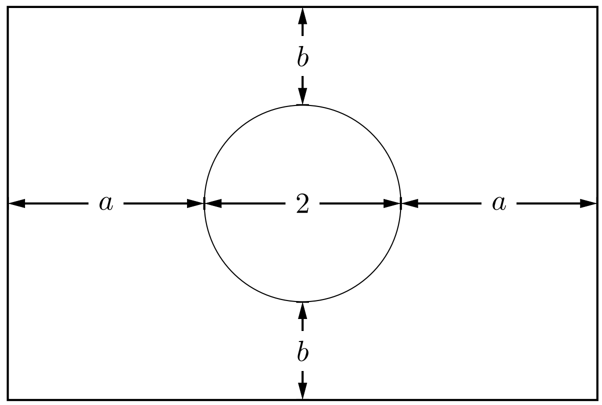
ii. Write an equation for the tangent line to the curve at the point $(1, 2)$.

Solution: We use point-slope form:

$$y - 2 = -2(x - 1) \quad \text{so} \quad y = 2 - 2(x - 1) \quad \text{or} \quad y = 4 - 2x.$$

4. [10 points] A landscaper is designing a rectangular garden surrounding a circular fountain in the middle.

- The diameter of the fountain is 2 meters.
- The distance from the fountain to the eastern and western edges of the garden is a meters.
- The distance from the fountain to the northern and southern edges of the garden is b meters.
- The part of the garden **outside of the circular fountain** will be covered with exactly 300 square meters of grass.



a. [4 points] Write a formula for b in terms of a.

Solution: Observe that the radius of the fountain is 1 meter, so that its area is $\pi \cdot 1^2 = \pi$. The length of the rectangle is $2 + 2a$, and the width is $2 + 2b$, so the area of the rectangle is $(2 + 2a)(2 + 2b)$. The area of the part of the garden outside the circular fountain is the difference between these two numbers we have found: $(2 + 2a)(2 + 2b) - \pi$. We are told that this area is equal to 300 meters, and so we have an equation that we can use to solve for b:

$$(2 + 2a)(2 + 2b) - \pi = 300$$
$$2 + 2b = \frac{300 + \pi}{2 + 2a}$$
$$b = \frac{1}{2}\left(\frac{300 + \pi}{2 + 2a} - 2\right) = \frac{300 + \pi - 4a - 4}{4a - 4}.$$

b. [2 points] Write a formula for the function $P(a)$ which gives the rectangular perimeter of the garden in terms of a only.

Solution: The rectangular perimeter of the garden is equal to $2(2a + 2) + 2(2b + 2)$. We may use our equation $2 + 2b = \frac{300+\pi}{2+2a}$ to write this in terms of a:

$$P(a) = 2(2a + 2) + 2\left(\frac{300 + \pi}{2 + 2a}\right).$$

c. [4 points] In the context of this problem, what is the domain of $P(a)$?

Solution: The domain of $P(a)$ is constrained by the following two facts:

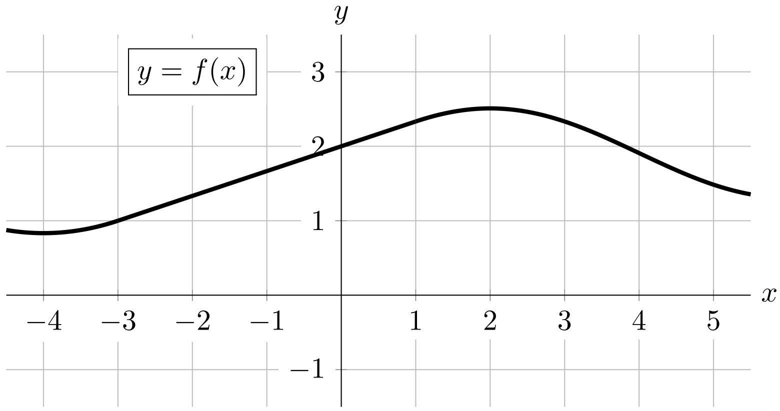
- a cannot be less than 0.
- b cannot be less than 0.

From the first fact, we see that the domain of $P(a)$ must begin with "[0." Let us express the second fact as a statement about a. Using the formula $b = \frac{1}{2}\left(\frac{300+\pi}{2+2a} - 2\right)$, we see that the second fact can be expressed as:

$$\begin{aligned} 0 &\leq \frac{1}{2}\left(\frac{300+\pi}{2+2a} - 2\right) \\ 2 &\leq \frac{300+\pi}{2+2a} \\ 4+4a &\leq 300+\pi \\ a &\leq \frac{300+\pi-4}{4}. \end{aligned}$$

Therefore, we see that the domain of $P(a)$ must end with "$\frac{300+\pi-4}{4}$]." Putting these together, we conclude that the domain of $P(a)$ is $\boxed{[0, \frac{300+\pi-4}{4}]}$.

5. [10 points] The graph of the function $f(x)$ is shown below.
Note that $f(x)$ is linear on the interval $(-3, 1)$.



a. [6 points] The function $g(x)$ is given by the equation

$$g(x) = \begin{cases} e^{px} & x \leq 0 \\ Cf(x) & x > 0 \end{cases}$$

where C and p are constants and f is as above. Find one pair of **exact** values for C and p such that $g(x)$ is differentiable, or write NONE if there are none. Be sure your work is clear.

Solution: We first must ensure that $g(x)$ is continuous at $x = 0$. We see that $\lim_{x\to 0^-} g(x) = e^{p\cdot 0}$ and $\lim_{x\to 0^+} g(x) = Cf(0)$, and so we solve:

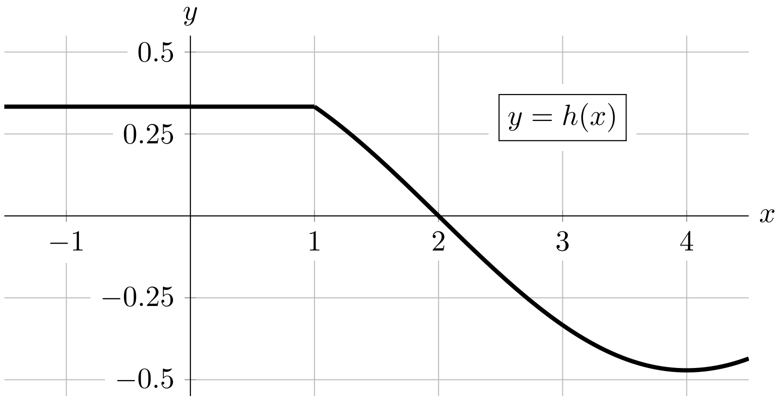
$$\begin{aligned} e^{p\cdot 0} &= Cf(0) \\ 1 &= C \cdot 2 \\ \frac{1}{2} &= C. \end{aligned}$$

We must next ensure that $\lim_{x\to 0^-} g'(x) = \lim_{x\to 0^+} g'(x)$. We see that $\lim_{x\to 0^-} g'(x) = pe^{p\cdot 0}$ and $\lim_{x\to 0^+} g'(x) = Cf'(0)$. Since $f(x)$ is linear on $(-3, 1)$, we compute its slope on the interval $[-3, 0]$ to find that $f'(0) = (2-1)/(0-(-3)) = \frac{1}{3}$. We now solve

$$\begin{aligned} pe^{p\cdot 0} &= Cf'(0) \\ p \cdot 1 &= \frac{1}{2} \cdot \frac{1}{3} \\ p &= \frac{1}{6}. \end{aligned}$$

We conclude that $\boxed{C = \frac{1}{2},\ p = \frac{1}{6}.}$

Part of the graph of the function $h(x)$ is shown below.



Note that $h(4) = -\frac{\sqrt{2}}{3}$.

b. [2 points] Complete the following sentence.

Because the function $h(x)$ satisfies the hypotheses of the mean value theorem on the interval $[2, 4]$, there must be some point c with $2 \le c \le 4$ such that…

Solution: … $h'(c) = \frac{h(4)-h(2)}{4-2} = -\frac{\sqrt{2}}{6}$.

c. [2 points] On which of the following intervals does $h(x)$ satisfy the hypotheses of the mean value theorem? List all correct answers, or write NONE.

$[-1, 0]$ $[0, 3]$ $[1, 4]$

Solution: The hypotheses of the mean value theorem for a function $h(x)$ on an interval $[a, b]$ are:

- Our function $h(x)$ is continuous on the closed interval $[a, b]$.
- Our function $h(x)$ is differentiable on the open interval (a, b).

-These are true for $h(x)$ on $[-1, 0]$ because $h(x)$ is constant on this interval.
-These are not true for $h(x)$ on $[0, 3]$ because we see from the graph that $h(x)$ is not differentiable at $x = 1$, which is inside the open interval $(0, 3)$.
-These are true for $h(x)$ on $[1, 4]$, because we see from the graph that $h(x)$ is continuous on this interval, and the only point at which $h(x)$ is not differentiable is $x = 1$, which is *not* inside the open interval $(1, 4)$.
-We conclude that $h(x)$ satisfies the hypotheses of the mean value theorem on $\boxed{[-1, 0] \text{ and } [1, 4].}$

6. [10 points] A manufacturer is constructing a closed hollow cylindrical tank out of a metal that costs \$2 per square foot. (Note that the tank has both a bottom and a top made of this same metal.) The tank's top must also be coated with a chemical that costs \$5 per square foot. The manufacturer will spend exactly \$180 on the tank.

- Find the height and radius of the cylindrical tank, in feet, so that the tank has the maximum possible volume.
- What is the maximum volume in this case, in cubic feet?

In your solution, make sure to carefully define any variables and functions you use, use calculus to justify your answers, and show enough evidence that the values you find do in fact maximize the volume.
Please note that the explanation below is much more extensive than what was required to earn full credit on the problem.

Solution: Let h denote the height of the tank, in feet, and let r denote the radius of the tank, in feet. The total cost of the tank is:

$$\text{Cost of tank} = \text{Cost of bottom} + \text{Cost of side} + \text{Cost of top}.$$

- The bottom of the tank has surface area πr^2 square feet, and costs \$2 per square foot. Therefore Cost of bottom $= 2\pi r^2$ dollars.
- The side of the tank has surface area $2\pi rh$ square feet, and costs \$2 per square foot. Therefore Cost of side $= 4\pi rh$ dollars.
- The top of our tank has surface area πr^2, and costs $\$2 + \$5 = \$7$ per square foot. Therefore Cost of top $= 7\pi r^2$ dollars.

Since we are told the manufacturer will spend \$180 on the tank, we have

$$\begin{aligned}\text{Cost of tank} &= 2\pi r^2 + 4\pi rh + 7\pi r^2\\ 180 &= 9\pi r^2 + 4\pi rh.\end{aligned}$$

We can use this equation to solve for h in terms of r:

$$\begin{aligned}9\pi r^2 + 4\pi rh &= 180\\ h &= \frac{180 - 9\pi r^2}{4\pi r} = \frac{45}{\pi} r^{-1} - \frac{9}{4} r.\end{aligned}$$

We want to maximize the volume of the cylinder, so let $V(r)$ denote the volume of the cylinder, in cubic feet, as a function of r. Then

$$\begin{aligned}V(r) &= \pi r^2 h\\ &= \pi r^2 \left(\frac{45}{\pi} r^{-1} - \frac{9}{4} r\right)\\ &= 45r - \frac{9\pi}{4} r^3.\end{aligned}$$

The domain of the function $V(r)$ is constrained by the following two facts:

- r cannot be less than 0.
- h cannote be less than 0.

From the first fact, we see that the domain of $V(r)$ must begin with "[0."

Solution: (Continued)
Let us express the second fact as a statement about r. Using the formula $h = \frac{45}{\pi}r^{-1} - \frac{9}{4}r$, we see that the second fact can be expressed as:

$$\begin{aligned} 0 &\le \frac{45}{\pi}r^{-1} - \frac{9}{4}r \\ \frac{9}{4}r &\le \frac{45}{\pi}r^{-1} \\ \frac{9}{4}r^2 &\le \frac{45}{\pi} \\ -\sqrt{\frac{60}{3\pi}} \le r &\le \sqrt{\frac{60}{3\pi}}. \end{aligned}$$

Since we already know r cannot be less than 0, this second inequality tells us that the domain of $V(r)$ must end with "$\sqrt{\frac{60}{3\pi}}\,]$." Putting these together, we conclude that the domain of $V(r)$ is $\left[0, \sqrt{\frac{60}{3\pi}}\right]$. We differentiate:

$$\begin{aligned} V(r) &= 45r - \frac{9\pi}{4}r^3 \\ V'(r) &= 45 - \frac{27\pi}{4}r^2. \end{aligned}$$

We find the r-values that make this derivative equal to 0:

$$\begin{aligned} 0 &= 45 - \frac{27\pi}{4}r^2 \\ r^2 &= \frac{45 \cdot 4}{27\pi} \\ r &= \pm\sqrt{\frac{20}{3\pi}}. \end{aligned}$$

Only one of these values, $\sqrt{\frac{20}{3\pi}}$, is inside the domain $\left[0, \sqrt{\frac{60}{3\pi}}\right]$.
To use the Second Derivative Test, we compute the second derivative

$$V''(r) = -\frac{27\pi}{2}r,$$

which is negative when r is positive; in particular $V''(r) < 0$ when $r = \sqrt{\frac{20}{3\pi}}$. Therefore $V(r)$ has a local maximum at $r = \sqrt{\frac{20}{3\pi}}$. Since this is the *only* critical point of $V(r)$ on the interval $\left[0, \sqrt{\frac{60}{3\pi}}\right]$, we conclude that $V(r)$ has a *global* maximum at $r = \sqrt{\frac{20}{3\pi}}$ on this interval. Therefore the maximum volume of the tank occurs when

$$\boxed{r = \sqrt{\frac{20}{3\pi}} \text{ feet,}} \quad \boxed{h = \frac{180 - 9\pi \cdot \frac{20}{3\pi}}{4\pi\sqrt{\frac{20}{3\pi}}} \text{ feet.}}$$

The maximum volume in this case is

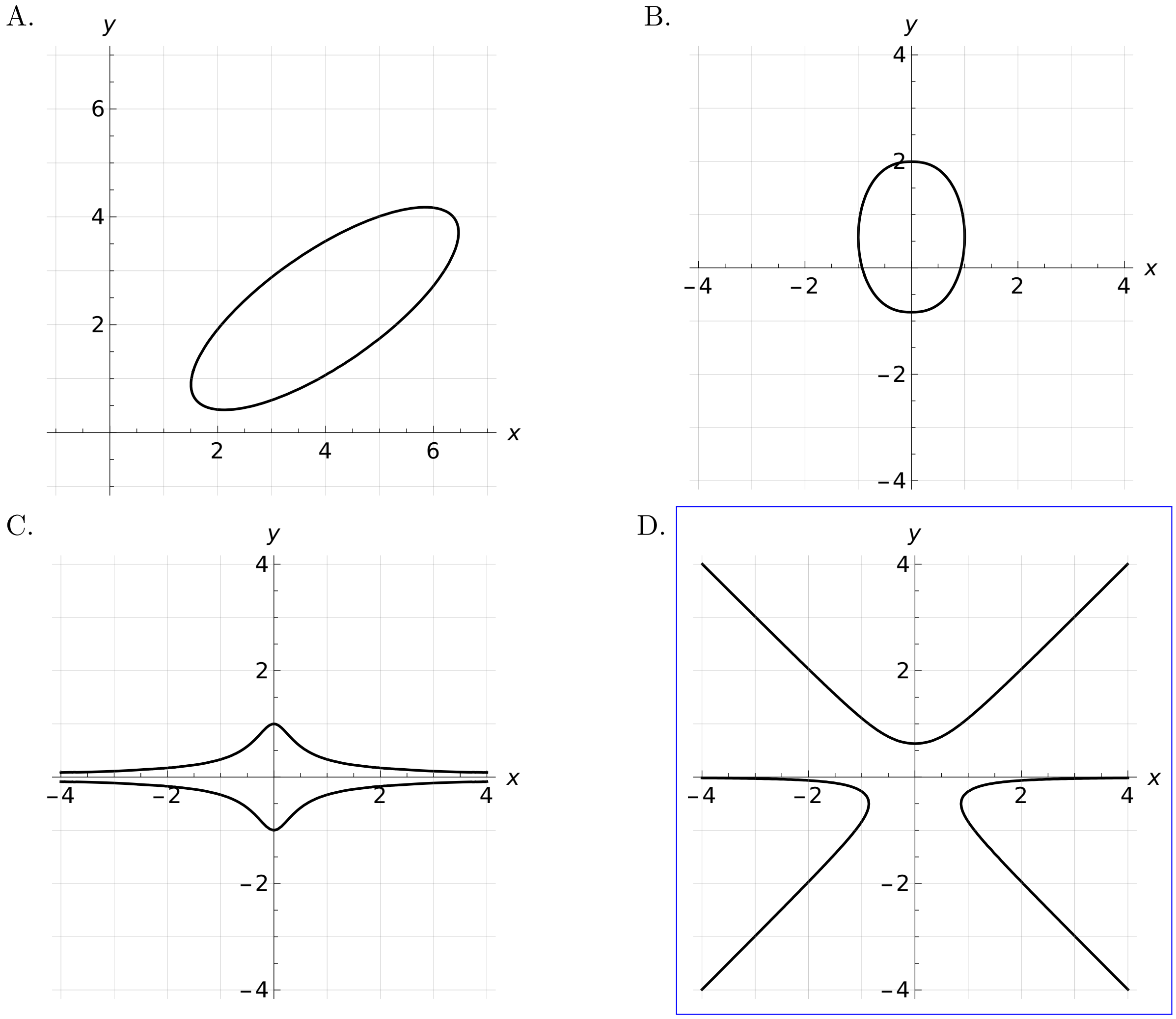
$$V\left(\sqrt{\frac{20}{3\pi}}\right) = \boxed{45\sqrt{\frac{20}{3\pi}} - \frac{9\pi}{4}\sqrt{\frac{20}{3\pi}}^{\,3} \text{ cubic feet.}}$$

7. [4 points] A curve $\mathcal{C}$ gives y as an implicit function of x and satisfies

$$\frac{dy}{dx} = \frac{2xy}{3y^2 - x^2} \quad \text{which can be factored and rewritten as} \quad \frac{dy}{dx} = \frac{2xy}{(\sqrt{3}y - x)(\sqrt{3}y + x)}.$$

One of the following graphs is the graph of the curve $\mathcal{C}$. Write the letter corresponding to that graph.

Hint: Look for horizontal and vertical tangent lines.



Solution: The formula $\frac{dy}{dx} = \frac{2xy}{(\sqrt{3}y - x)(\sqrt{3}y + x)}$ tells us that $\mathcal{C}$ has a horizontal tangent line only when $x = 0$ or $y = 0$, and $\mathcal{C}$ has a vertical tangent line only when $x = \sqrt{3}y$ or $x = -\sqrt{3}y$.

- Option A has horizontal tangent lines when neither $x = 0$ nor $y = 0$, and so cannot be the graph of $\mathcal{C}$.
- Option B has tangent lines that are not horizontal when $y = 0$, and so cannot be the graph of $\mathcal{C}$.
- The equations $x = \sqrt{3}y$ and $x = -\sqrt{3}y$ are equations for lines of slope $1/\sqrt{3}$ and $-1/\sqrt{3}$ passing through the origin. The graph of $\mathcal{C}$ therefore must have a vertical tangent line whenever it intersects one of these lines. Option C passes through the y-axis, and also has the x-axis as a horizontal asymptote. Therefore Option C must pass through every non-horizontal line, and in particular, the two lines we have just described. But Option C has no vertical tangent lines, and so we conclude it cannot be the graph of $\mathcal{C}$.
- We have eliminated every other option, and so we conclude that **Option D** is the graph of $\mathcal{C}$.

8. [11 points] Suppose $J(x)$ is a continuous function defined for all real numbers x. The **derivative** and **second derivative** of $J(x)$ are given by

$$J'(x) = \frac{x^2(x-1)}{\sqrt[3]{x+4}} \qquad \text{and} \qquad J''(x) = \frac{x(8x^2+31x-24)}{3\left(\sqrt[3]{x+4}\right)^4}.$$

a. [2 points] Find the x-coordinates of all critical points of $J(x)$. If there are none, write NONE.

Solution: The critical points of $J(x)$ occur when $J'(x) = 0$ or $J'(x)$ does not exist. We see that $J'(x) = 0$ when $x = 0$ and $x = 1$, and we see that $J'(x)$ does not exist when $x = -4$. We conclude that the critical points of $J(x)$ occur at $\boxed{x = 0, 1, \text{ and } -4}$.

Throughout parts **b.** and **c.** below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

b. [5 points] Find the x-coordinates of

i. all local minima of $J(x)$ and

ii. all local maxima of $J(x)$.

If there are none of a particular type, write NONE.

Solution: We use the First Derivative Test. Observe that x^2 is always positive unless $x = 0$.

- On the interval $(-\infty, -4)$, we see that $x - 1$ is negative and $\sqrt[3]{x+4}$ is also negative, and so $J'(x)$ is positive on this interval.
- On the interval $(-4, 0)$, we see that $x - 1$ is negative and $\sqrt[3]{x+4}$ is positive, and so $J'(x)$ is negative on this interval.
- On the interval $(0, 1)$, we see that $x - 1$ is negative and $\sqrt[3]{x+4}$ is positive, and so $J'(x)$ is negative on this interval.
- On the interval $(1, \infty)$, we see that $x - 1$ is positive and $\sqrt[3]{x+4}$ is also positive, and so $J'(x)$ is positive on this interval.

Since $J'(x)$ is positive to the left of $x = -4$ and negative to the right, we see $J'(x)$ has a $\boxed{\text{local maximum at } x = -4.}$ Since $J'(x)$ is negative to the left of $x = 0$ and negative to the right, we see that $J'(x)$ has neither a local maximum nor minimum at $x = 0$. Since $J'(x)$ is negative to the left of $x = 1$ and positive to the right, we see that $J'(x)$ has a $\boxed{\text{local minimum at } x = 1.}$

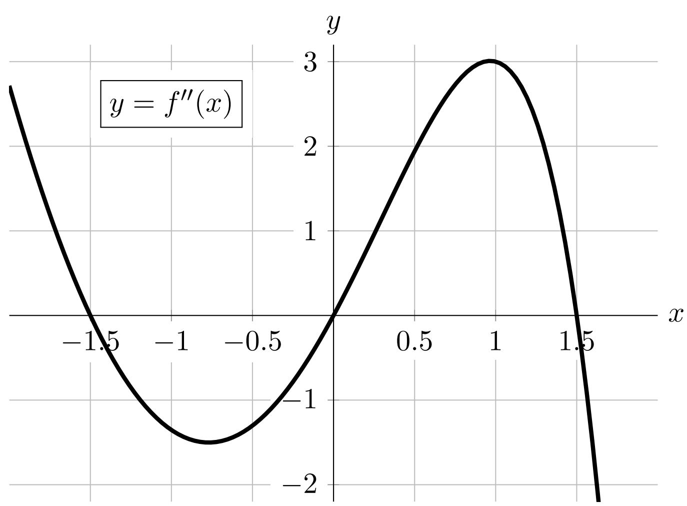
c. [4 points] The polynomial $8x^2+31x-24$ (from the numerator of $J''(x)$) has two zeroes a and b, where $a \approx -4.54$ and $b \approx 0.66$. How many inflection points does the function $J(x)$ have? Remember to justify your answer. *Hint: What does the graph of* $8x^2+31x-24$ *look like?*

Solution: The inflection points of $J(x)$ can only occur when $J''(x)=0$ or $J''(x)$ does not exist. We see that $J''(x)=0$ when $x=0,a,$ and b, and we see that $J''(x)$ does not exist when $x=-4$. We consider the signs of $J''(x)$ on the intervals between these points. Observe that $\left(\sqrt[3]{x+4}\right)^4$ is always positive unless $x=-4$. Also observe that the graph of $8x^2+31x-24$ is a concave up parabola, which tells us that $8x^2+31x-24$ is only negative when $a<x<b$.

- On the interval $(-\infty,a)$, we see that x is negative and $8x^2+31x-24$ is positive, and so $J''(x)$ is negative.
- On the interval $(a,-4)$, we see that x is negative and $8x^2+31x-24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(-4,0)$, we see that x is negative and $8x^2+31x-24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(0,b)$, we see that x is positive and $8x^2+31x-24$ is negative, and so $J''(x)$ is negative.
- On the interval (b,∞), we see that x is positive and $8x^2+31x-24$ is also positive, and so $J''(x)$ is positive.

We observe that $J''(x)$ changes sign only at $x=a,0,$ and b, and so we conclude that $J''(x)$ has 3 inflection points.

9. [16 points] We consider a function $f(x)$ defined for all real numbers. We suppose that the first and second derivatives $f'(x)$ and $f''(x)$ are also defined for all real numbers. Below we show the graph of the **second derivative** of f. You may assume that $f''(x)$ is decreasing outside of the region shown.



a. [3 points] Find or estimate the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE.

Solution: The inflection points of $f(x)$ occur when $f''(x)$ changes sign. Therefore the x-coordinates of the inflection points of $f(x)$ are $x=-1.5,0,$ and 1.5.

b. [3 points] Find or estimate the x-coordinates of all inflection points of $f'(x)$. If there are none, write NONE.

Solution: The inflection points of $f'(x)$ occur when $f'''(x)$ changes sign. Since $f'''(x)$ is the first derivative of $f''(x)$, this means that we are looking for points where the graph changes from increasing to decreasing, or vice versa. These occur at $\boxed{x \approx -0.75 \text{ and } x \approx 0.9.}$

c. [1 point] Suppose that $f'(0) = 5$. How many critical points does f have?

Solution: By looking at the graph, we see that $f'(x)$ is increasing on the intervals $(-\infty, -1.5)$ and $(0, 1.5)$, and is decreasing on the intervals $(-1.5, 0)$ and $(1.5, 0)$. This tells us that $f'(x)$ has a local minimum at $x = 0$ and local maxima at $x = -1.5$ and $x = 1.5$. Since $f'(0) = 5$, this means that $f'(x)$ cannot be equal to 0 on the interval $(-1.5, 1.5)$, because it is always positive on this interval.

Since $f''(x)$ is decreasing and positive on the interval $(-\infty, -1.5)$, we see that $f'(x)$ is increasing and concave down on this interval. Therefore $f'(x)$ diverges to $-\infty$ as $x \to -\infty$. Since $f'(-1.5)$ is positive, we conclude that $f'(x)$ must cross the x-axis *once* on the interval $(-\infty, -1.5)$. Similarly, since $f''(x)$ is decreasing and negative on the interval $(1.5, \infty)$, we see that $f'(x)$ is decreasing and concave down on this interval. Therefore $f'(x)$ diverges to $-\infty$ as $x \to \infty$. Since $f'(1.5)$ is positive, we conclude that $f'(x)$ must cross the x-axis *once* on the interval $(1.5, \infty)$.

We conclude that $f'(x)$ is equal to 0 at exactly two x-values, and so the function f has $\boxed{\text{2 critical points.}}$

For parts **d.-f.** below, suppose that $f'(1) = 6.8$ and $f(1) = 4$.

d. [4 points] Let $Q(x)$ be the quadratic approximation of $f(x)$ near $x = 1$. Find a formula for $Q(x)$.

Solution: We have

$$Q(x) = \frac{f''(1)}{2}(x-1)^2 + f'(1)(x-1) + f(1)$$
$$= \boxed{\frac{3}{2}(x-1)^2 + 6.8(x-1) + 4.}$$

e. [2 points] Is the linear approximation of $f(x)$ near $x = 1$ an overestimate or an underestimate of $f(x)$ for values of x near 1? Explain your reasoning.

Solution: Since $f''(1) > 0$, it follows that the linear approximation of $f(x)$ near $x = 1$ is an $\boxed{\text{underestimate}}$ of $f(x)$ for values of x near 1.

f. [3 points] Let $L(x)$ be the linear approximation of $f'(x)$ (the derivative of f) near $x = 1$. Find a formula for $L(x)$.

Solution: We have

$$L(x) = f''(1)(x-1) + f'(1)$$
$$= \boxed{3(x-1) + 6.8.}$$

10. [10 points] Consider a continuous function $f(x)$, and suppose that $f(x)$ and its first derivative $f'(x)$ are differentiable everywhere. Suppose we know the following information about $f(x)$ and its first and second derivatives.

- On the interval $(-\infty, -2)$, we have $f(x) = 2^{-x}$.
- $\lim_{x\to\infty} f(x) = 6$.
- $f(2) = -5$, $f(3) = 7$, and $f(4) = 8$.
- $f'(x)$ is equal to 0 at $x = -1$, 2, 4, and not at any other x-values.
- $f''(x) < 0$ on the intervals $-1 < x < 0$ and $3 < x < 5$, and not on any other interval.

For each part below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [5 points] Find the x-coordinates of

i. the global minimum(s) of $f(x)$ on $[3, \infty)$ and

ii. the global maximum(s) of $f(x)$ on $[3, \infty)$.

If there are none of a particular type, write NONE. If there is not enough information to find a desired x-coordinate, write NEI.

Solution: The only critical point of $f(x)$ on this interval is at $x = 4$.

We have $f(3) = 7$, $f(4) = 8$, and $\lim_{x\to\infty} f(x) = 6$.

We conclude that **the global maximum occurs at $x = 4$.**

Since $\lim_{x\to\infty} f(x) < f(3)$, we conclude that **there is no global minimum.**

b. [5 points] Find the x-coordinates of

i. the global minimum(s) of $f(x)$ on $(-\infty, \infty)$ and

ii. the global maximum(s) of $f(x)$ on $(-\infty, \infty)$.

If there are none of a particular type, write NONE. If there is not enough information to find a desired x-coordinate, write NEI.

Solution: Since 2^{-x} diverges to ∞ as $x \to -\infty$, we conclude that **there is no global maximum.**

We now must find the global minimum. We know that $f(2) = -5$ and $\lim_{x\to\infty} f(x) = 6$, so there must be a global minimum somewhere. It can only occur at a critical point of $f(x)$, and since $f(4) = 8$, it does not occur at $x = 4$. We now must decide whether the global minimum occurs at $x = 2$ or $x = -1$.

Since $f''(x) < 0$ on the interval $-1 < x < 0$, we see that $f'(x)$ is decreasing on this interval. Since $f'(-1) = 0$, that means that $f'(x)$ must be negative on this interval. That means that $f(x)$ is decreasing on this interval. That means that $f(-0.5) < f(-1)$, and so the function $f(x)$ cannot have a global minimum at $x = -1$.
We conclude that **the global minimum occurs at $x = 2$.**