Math 115 — Second Midterm — March 22, 2022

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ______________________

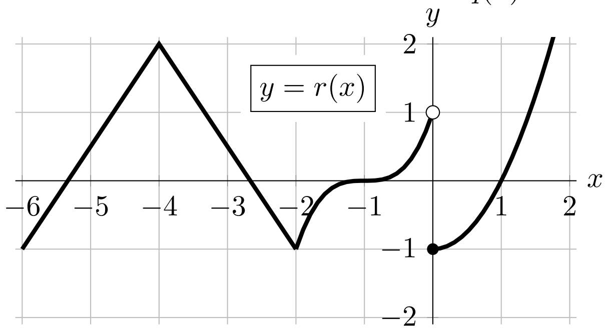
Instructor Name: ____ Section #: ______________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	7	
3	10	
4	8	
5	15	

Problem	Points	Score
6	9	
7	5	
8	8	
9	11	
10	8	
Total	90	

1. [9 points] A portion of a graph of the function $r(x)$, whose domain is $(-\infty, \infty)$ is shown below to the left. The function $r(x)$ is linear on the intervals $[-6, -4]$ and $[-4, -2]$. A table of values for a differentiable and invertible function $q(x)$ and its derivative $q'(x)$ are shown below to the right.



x	-3	-2	-1	0	1	2	3
$q(x)$	14	10	3	2	-5	-6	-15
$q'(x)$	-10	-12	-4	0	-2	-5	-6

Find the **exact** values of the quantities in parts **a.-d.**, whenever possible. Write NEI if there is not enough information to do so, or write DNE if the value does not exist. Your answers should not include the letters q or r but you do not need to simplify your numerical answers. Show your work.

a. [1 point] Find $r'(-4)$.

Answer: $r'(-4) =$ ____________________

b. [2 points] Find $(q^{-1})'(-6)$.

Answer: $(q^{-1})'(-6) =$ ____________________

c. [3 points] Let $J(x) = e^{q(x)}$. Find $J'(1)$.

Answer: $J'(1) =$ ____________________

d. [3 points] Let $D(x) = r(x)q(2x + 4)$. Find $D'(-3)$.

Answer: $D'(-3) =$ ____________________

2. [7 points] A table of values for a differentiable and invertible function $q(x)$ and its derivative $q'(x)$ are shown below. Note that this is the same function q as on the previous page. However, you do not need your work or answers from the previous page to do this problem.

x	-3	-2	-1	0	1	2	3
$q(x)$	14	10	3	2	-5	-6	-15
$q'(x)$	-10	-12	-4	0	-2	-5	-6

Let $\mathcal{C}$ be the curve defined implicitly by the equation

$$xy^2 + \sin(2\pi q(x)) = 6e^{y-4} + 10.$$

a. [1 point] Exactly one of the following points (x, y) lies on the curve $\mathcal{C}$. Circle that <u>one</u> point.

$(-2, 1)$ $\qquad$ $(1, 4)$ $\qquad$ $(0, 4)$ $\qquad$ $(0, 10)$

b. [6 points] Find an equation for the tangent line to the curve $\mathcal{C}$ at the point you chose in part **a**. Make sure to show your work clearly.

Answer: $y =$ ________________________

3. [10 points] Suppose $B(x)$ is a continuous function defined for all real numbers x. The derivative and second derivative of $B(x)$ are given by

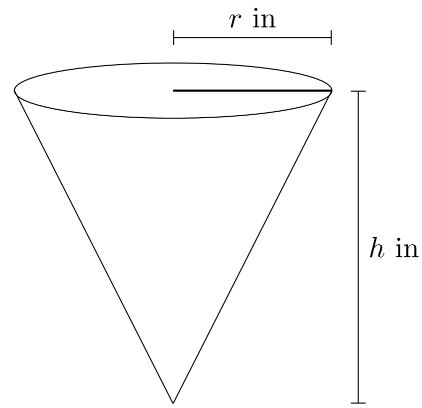
$$B'(x) = e^{-x}x(\sqrt[3]{x-2})^2 \quad \text{and} \quad B''(x) = \frac{-e^{-x}(x-3)(3x-2)}{3\sqrt[3]{x-2}}.$$

a. [5 points] Find the exact x-coordinates of all local minima and local maxima of $B(x)$. If there are none of a particular type, write NONE. You must use calculus to find and justify your answers. Be sure your conclusions are clearly stated and that you show enough evidence to support them.

Answer: Local min(s) at $x =$ ____________ and Local max(es) at $x =$ ____________

b. [5 points] Find the exact x-coordinates of all inflection points of $B(x)$, or write NONE if there are none. You must use calculus to find and justify your answers. Be sure your conclusions are clearly stated and that you show enough evidence to support them.

Answer: Inflection point(s) at $x =$ ____________________



4. [8 points]
Sunny and Tyrell own an ice cream shop together. They want to sell waffle cones in the usual shape of a cone, as shown on the right. The cost, in dollars, of a waffle cone with radius r inches and height h inches is

$$\frac{r}{2}\left(\sqrt{h^2+r^2}\right).$$

Sunny and Tyrell want to spend exactly \$5 on a waffle cone that can fit the most ice cream (i.e has the largest volume).

Note that the volume of a cone of radius r and height h is $\dfrac{\pi r^2 h}{3}$.

a. [3 points] Write a formula for h in terms of r if the cone costs \$5.

Answer: $h =$ ____________________

b. [2 points] Write a formula for the function $V(r)$ which gives the volume, in cubic inches, of an ice cream cone that costs \$5 in terms of r only. *Your formula should not include the letter h.*

Answer: $V(r) =$ ____________________

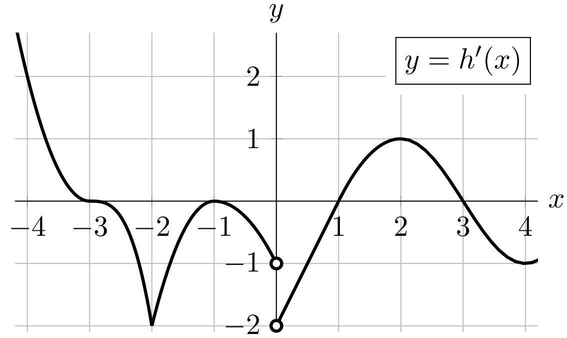
c. [3 points] What is the domain of $V(r)$ in the context of this problem?

Answer: ____________________

5. [15 points]
Shown on the right is the graph of $h'(x)$, the **derivative** of a function $h(x)$. Assume that h is continuous on its entire domain $(-\infty, \infty)$.

Use this graph to answer the questions below.

You may also use the fact that $\boxed{h(-4) = 5}$.



a. [3 points] Find the linear approximation $L(x)$ of $h(x)$ near $x = -4$, and use your formula to approximate $h(-3.9)$.

Answer: $L(x) =$ __________________ and $h(-3.9) \approx$ __________________

b. [2 points] Is the estimate of $h(-3.9)$ in part **a.** an overestimate or underestimate of the actual value, or is there not enough information to decide? Briefly explain your reasoning.

Circle one: OVERESTIMATE UNDERESTIMATE NOT ENOUGH INFORMATION

Brief explanation:

For each question below, circle **all** correct choices. You do not need to justify your answers.

c. [2 points] At which of the following values of x does $h(x)$ have a critical point?

$x = -2$ $x = -1$ $x = 0$ $x = 2$ $x = 3$ NONE OF THESE

d. [2 points] At which of the following values of x does $h(x)$ have a local maximum?

$x = -1$ $x = 0$ $x = 1$ $x = 2$ $x = 3$ NONE OF THESE

e. [2 points] At which of the following values of x does $h(x)$ have an inflection point?

$x = -3$ $x = -2$ $x = -1$ $x = 0$ $x = 2$ NONE OF THESE

f. [2 points] If $g(x) = h'(x)$, on which of the following interval(s) does $g(x)$ satisfy the hypotheses of the Mean Value Theorem?

$[-4, -1]$ $[-1, 2]$ $[1, 3]$ $[2, 4]$ NONE OF THESE

g. [2 points]. If $g(x) = h'(x)$, on which of the following interval(s) does $g(x)$ satisfy the conclusion of the Mean Value Theorem?

$[-4, -1]$ $[-1, 2]$ $[1, 3]$ $[2, 4]$ NONE OF THESE

6. [9 points] The Loads-of-Oats company is designing a new cylindrical container for their steel-cut oats. The company specifies that

- the height of the cylinder and four times the radius of the cylinder should sum to 18 inches
- the radius of the cylinder will be at least 1 inch, and
- the height of the cylinder will be at least 2 inches.

a. [2 points] What is the largest possible radius of such a cylindrical container?

Answer: ______________________

b. [7 points] Find the height and radius of such a cylindrical container, in inches, that maximize the volume of the container.

In your solution, make sure to carefully define any variables and functions you use. Use calculus to justify your answers, and show enough evidence that the values you find do in fact maximize the volume.

Answer: height = ______________ and radius = ______________

7. [5 points] A function $g(x)$ is given by the following formula, where K and M are constants:

$$g(x) = \begin{cases} Ke^{-x+5} & x \leq 5 \\ M + \sqrt{x+4} & x > 5. \end{cases}$$

Find all values of K and M so that $g(x)$ is differentiable on $(-\infty, \infty)$. Write NONE if there are no such values. You do not need to simplify your answers, but show your work clearly.

Answer: $K =$ ________________ and $M =$ ________________

8. [8 points] Prairie dogs named Paws and Dot have been hard at work digging a tunnel. Consider the functions L and C defined as follows:

- $L(w)$ is the length of the tunnel, in feet, when w pounds of dirt have been removed.
- $C(w)$ is the total number of Calories the prairie dogs have burned digging their tunnel when they have removed a total of w pounds of dirt for their tunnel.

The functions $L(w)$ and $C(w)$ are both invertible and differentiable.

a. [4 points] Complete the sentence below to give a practical interpretation of the equation

$$(L^{-1})'(10) = 24.$$

In order to increase the length of the tunnel from 10 *feet to* 10.25 *feet, ...*

b. [4 points]

i. Which of the following expressions gives the length, in feet, of the prairie dog tunnel when the prairie dogs have burned a total of x Calories digging? Circle the one correct expression.

$C(L^{-1}(x))$ $\qquad$ $C^{-1}(L(x))$ $\qquad$ $L(C^{-1}(x))$ $\qquad$ $L^{-1}(C(x))$

ii. Use the answer you selected in part i to find an expression for the instantaneous rate of change of the length of the prairie dog tunnel, in feet per calorie, when the prairie dogs have burned a total of 2000 calories digging.
Simplify as much as possible. Note that your final answer may involve the function names L, L^{-1}, L', C, C^{-1}, *and* C' *but should not involve the function names* $\left(L^{-1}\right)'$ *or* $\left(C^{-1}\right)'$.

Answer: ______________________________

9. [11 points] A continuous function $w(x)$ and its derivative $w'(x)$ are given by

$$w(x) = \begin{cases} x^2(3x^2+10x-9) & x \le 1 \\ -2\ln(3x-2)+4 & x > 1 \end{cases} \quad \text{and} \quad w'(x) = \begin{cases} 6x(x+3)(2x-1) & x < 1 \\ \dfrac{-6}{3x-2} & x > 1. \end{cases}$$

a. [2 points] Find the x-coordinates of all critical points of $w(x)$. If there are none, write NONE. You do not need to justify your answer.

Answer: Critical point(s) at $x =$ ______________________________

For each part below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

b. [4 points] Find the x-coordinates of all global minima and global maxima of $w(x)$ **on the interval** $(-\infty, 0)$. If there are none of a particular type, write NONE.

Answer: Global min(s) at $x =$ ____________ and Global max(es) at $x =$ ____________

c. [5 points] Find the x-coordinates of all global minima and global maxima of $w(x)$ **on the interval** $\left[-1, \frac{e+2}{3}\right]$. If there are none of a particular type, write NONE.
In case it is useful, note that $1 < \frac{e+2}{3} < 2$.

Answer: Global min(s) at $x =$ ____________ and Global max(es) at $x =$ ____________

10. [8 points] Some information about the derivative $p'(x)$ and the second derivative $p''(x)$ of a function $p(x)$ is provided in the table below.

x	-4	-3	-2	-1	0	1	2
$p'(x)$	1	0	-2	0	-1	0	2
$p''(x)$	-1	0	0	0	0	2	1

Assume that

- $p''(x)$ is defined and continuous on the interval $(-\infty, \infty)$ and
- the values of both $p'(x)$ and $p''(x)$ are strictly positive or strictly negative between consecutive table entries.

For each question below, circle **all** correct choices. You do not need to justify your answers.

a. [2 points] On which of the following intervals must $p(x)$ be always concave up?

$-4 < x < -3$ $\quad$ $-3 < x < -2$ $\quad$ $-2 < x < -1$

$-1 < x < 0$ $\quad$ $0 < x < 1$ $\quad$ $1 < x < 2$ $\quad$ NONE OF THESE

b. [2 points] At which of the following values of x must $p(x)$ have a local minimum?

$x = -3$ $\quad$ $x = -2$ $\quad$ $x = -1$ $\quad$ $x = 0$ $\quad$ $x = 1$ $\quad$ NONE OF THESE

c. [2 points] At which of the following values of x must $p(x)$ have an inflection point?

$x = -3$ $\quad$ $x = -2$ $\quad$ $x = -1$ $\quad$ $x = 0$ $\quad$ $x = 1$ $\quad$ NONE OF THESE

d. [2 points] At which value(s) of x does $p(x)$ attain a global maximum on the interval $[-4, 0]$?

$x = -4$ $\quad$ $x = -3$ $\quad$ $x = -2$ $\quad$ $x = -1$ $\quad$ $x = 0$

NONE OF THESE $\quad$ CANNOT BE DETERMINED