Math 115 — First Midterm — February 8, 2022

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 8 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. (You may not use any other paper.) Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	10	
2	8	
3	7	
4	9	

Problem	Points	Score
5	8	
6	5	
7	7	
8	6	
Total	60	

1. [10 points] Machiko and Nico are learning to sail on a river that runs east-west, and their friend Pennie is walking slowly on a sidewalk parallel to the river, near a bridge across the river. Functions M, N, and P are defined as follows. At the time t minutes after 10:00am,

- $M(t)$ is the position of Machiko's boat,
- $N(t)$ is the position of Nico's boat, and
- $P(t)$ is the position of Pennie.

All three positions are measured in yards east of the bridge.

The table on the right provides some values of these functions. Note that unknown values are shown as "?" in the table.

t (minutes)	1	3	6	7	9
$M(t)$ (yards)	?	24	36	?	?
$N(t)$ (yards)	50	2	?	?	?
$P(t)$ (yards)	7	21	45	40	8

a. [2 points] Find a formula for $M(t)$ assuming that $M(t)$ is a linear function.

Solution: Using the given points $(3, 24)$ and $(6, 36)$, the slope of $M(t)$ is $(36 - 24)/(6 - 3) = 12/3 = 4$. Therefore, $M(t) - 24 = 4(t - 3)$ or $M(t) = 24 + 4(t - 3) = 12 + 4t$.

Answer: $M(t) =$ $24 + 4(t - 3) = 12 + 4t$

b. [2 points] Find a formula for $N(t)$ assuming that $N(t)$ is an exponential function.

Solution: Since N is exponential, we have $N(t) = ab^t$, where we can solve for a and b using the given points. We must have $2 = 50b^2$ so $b^2 = \frac{1}{25}$ and hence $b = \frac{1}{5}$. Using the point $(1, 50)$ gives the equation $ab = 50$ so $a = 250$. Therefore, $N(t) = 250(1/5)^t$.

Answer: $N(t) =$ $250(1/5)^t$

c. [2 points] What is Pennie's average velocity between 10:03am and 10:07am? (Include units.)

Solution: $\dfrac{P(7) - P(3)}{7 - 3} = \dfrac{40 - 21}{4} = \dfrac{19}{4}$

Answer: $\dfrac{40 - 21}{7 - 3} = \dfrac{19}{4} = 4.75$ **yards/minute**

d. [2 points] Estimate $P'(8)$.

Solution: $P'(8) \approx \dfrac{P(9) - P(7)}{9 - 7} = \dfrac{8 - 40}{9 - 7} = \dfrac{-32}{2} = -16$

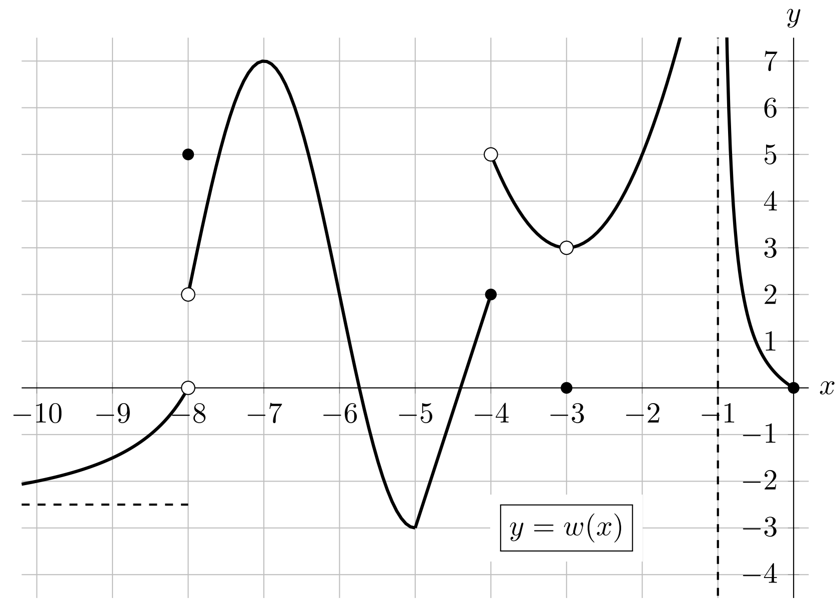
Answer: $P'(8) \approx$ $\dfrac{8 - 40}{9 - 7} = \dfrac{-32}{2} = -16$

e. [2 points] Suppose we also know that Pennie starts at the bridge at 10:00am, walks 50 yards directly east, then turns around and walks directly west back toward the bridge, without changing direction at any other time. For which of the following time intervals could $P'(t)$ be negative for *some* value of t in that interval? Circle all correct choices.

Solution: Note that Pennie must turn around either between 10:03 and 10:07 or between 10:06 and 10:07, but we cannot determine which from the given information. Hence, $P'(t)$ *must* be negative for some value of t in the intervals $[6, 7]$ and $[7, 9]$ and *could* be negative for some value of t in the interval $[3, 6]$.

$[1, 3]$ $\quad$ **$[3, 6]$** $\quad$ **$[6, 7]$** $\quad$ **$[7, 9]$** $\quad$ NONE OF THESE

2. [8 points] Below is a portion of a graph of an **even** function $w(x)$. Note that $w(x)$ has a vertical asymptote at $x = -1$, has a horizontal asymptote at $y = -2.5$, and is linear on $[-5, -4]$.



Evaluate each of the given quantities. If the value does not represent a real number (including the case of limits that diverge to ∞ and $-\infty$), write DNE. You do not need to show work in this problem. Give your answers in exact form.

a. [1 point] $\lim_{p\to -4^+} w(p)$

Answer: 5

b. [1 point] $\lim_{x\to -8} w(x)$

Answer: **DNE**

c. [2 points] $\lim_{h\to -1} w(-2+h)$

Solution: For h close to but not equal to -1, the quantity $-2+h$ is close to but not equal to -3. So, as $h \to -1$, we see from the graph that the value of $w(-2+h)$ approaches 3.

Answer: 3

d. [2 points] $\lim_{x\to \infty} w(x)$

Solution: Since w is an even function, the graph of w is symmetric across the vertical axis, so there is a horizontal asymptote on the right at $y = -2.5$. Alternatively, since w is even, we know $w(x) = w(-x)$ for all x in the domain of w, so $\lim_{x\to\infty} w(x) = \lim_{x\to\infty} w(-x) = -2.5$.

Answer: -2.5

e. [2 points] $\lim_{h\to 0} \left((3-h)^2 + \frac{w(-4.5+h) - w(-4.5)}{h} \right)$

Solution:

$$\lim_{h\to 0} \left((3-h)^2 + \frac{w(-4.5+h) - w(-4.5)}{h} \right) = \lim_{h\to 0} \left((3-h)^2\right) + w'(-4.5) = (3-0)^2 + 5 = 14.$$

Answer: 14

3. [7 points] Iggy the young inchworm starts crawling along a tree branch and eating leaves on the tree so that he can prepare to turn into a moth.

- Let $C(t)$ be the distance Iggy has crawled, in inches, after t minutes.
- Let $E(t)$ be the amount of leaves Iggy has eaten, in milligrams (mg), after t minutes of crawling.

The functions $C(t)$ and $E(t)$ are both invertible and differentiable.

a. [4 points] Find a mathematical equation for each of the statements below using the functions C, E, their inverses, and/or their derivatives.

i. Iggy has eaten 5 mg of leaves after crawling for 2 minutes.

Answer: $P^{-1}(5) = 2$ **or** $P(2) = 5$

ii. After crawling 10 inches, Iggy has eaten three times the amount of leaves as he had after crawling 6 inches.

Answer: $E\left(C^{-1}(10)\right) = 3E\left(C^{-1}(6)\right)$

iii. At 3 minutes of crawling, Iggy's instantaneous velocity is 14 inches per minute.

Answer: $C'(3) = 14$

b. [3 points] Complete the following sentence to give a practical interpretation of the equation

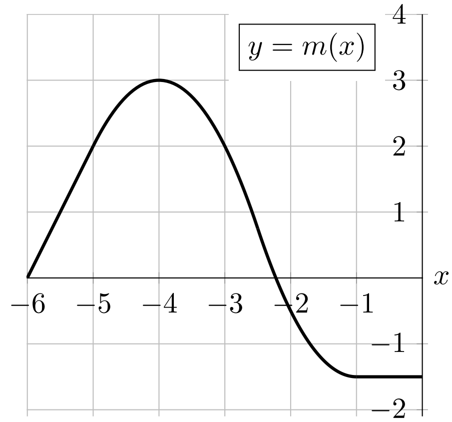
$$E'(3) = 4.$$

If Iggy eats leaves while crawling for 2 minutes and 30 seconds rather than for 3 minutes, then...

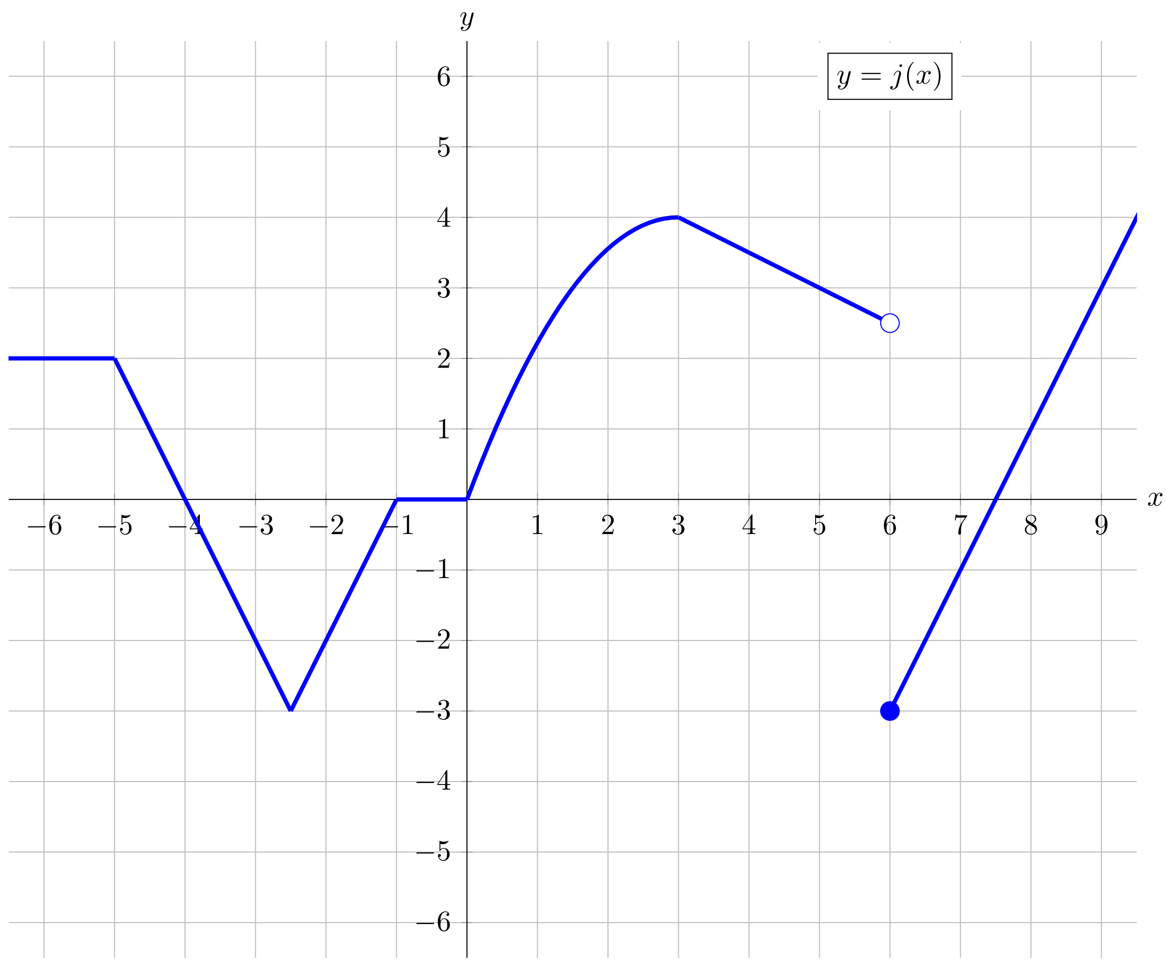
Solution: Iggy would eat about 2 mg less of leaves.

4. [9 points] On the axes provided below, sketch the graph of a single function $j(x)$ that satisfies all of the following conditions.

- The domain of the function $j(x)$ includes $-6 < x < 9$.
- On $-6 < x < 0$, the function $j(x)$ is the derivative of the function $m(x)$, which is shown in the graph to the right. Note that $m(x)$ is linear for $-6 < x < -5$ and is constant for $-1 < x < 0$.



- $j(x)$ is continuous on $0 < x < 5$.
- $j(x)$ is increasing and concave down on $0 < x < 3$.
- The average rate of change on $[3, 5]$ is $-\frac{1}{2}$.
- $\lim_{x \to 6} j(x)$ does not exist.
- $j(6) = -3$.
- The instantaneous rate of change of $j(x)$ at $x = 8$ is 2.



5. [8 points] The function $k(x)$ is given by the following formula, where A and B are positive constants:

$$k(x) = \begin{cases} 3 + e^{x-1} & x \le 1 \\ \dfrac{2x^2+5x+1}{Ax^2+1} & 1 < x < 2 \\ \ln(Bx) + 3 & x \ge 2. \end{cases}$$

a. [2 points] Evaluate each of the expressions below. If a limit does not exist, including if it diverges to ∞ or $-\infty$, write DNE. You do not need to show work.

$\lim_{x\to-\infty} k(x)$

Solution: $\lim_{x\to-\infty} k(x) = \lim_{x\to-\infty}(3+e^{x-1}) = 3$

Answer: 3

$\lim_{x\to\infty} k(x)$

Solution: $\lim_{x\to\infty} k(x) = \lim_{x\to\infty}(\ln(Bx)+3) = \infty$ (DNE)

Answer: **DNE**

b. [2 points] Find all horizontal and vertical asymptotes of $k(x)$ or write NONE if there are none.

Solution: By part a., the line $y = 3$ is a horizontal asymptote, and there are no others.

- The first piece of $k(x)$ has no vertical asymptotes since it is a shifted exponential.
- The rational piece of $k(x)$ has denominator that is always positive so there are no vertical asymptotes coming from that piece either.
- The function $\ln(Bx) + 3$ has a vertical asymptote at $x = 0$, but that value of x is not part of the relevant domain for the third piece of $k(x)$.

Hence, $k(x)$ has no vertical asymptotes.

Answer: Horizontal: $y = 3$ Vertical: **NONE**

c. [4 points] Find all values of A and B so that

- $k(x)$ is continuous at $x = 1$ and also
- $k(x)$ is continuous at $x = 2$.

Write NONE if there are no such values. Show your work.

Solution: For continuity at $x = 1$, we consider left and right limits at 1.
Note that $\lim_{x\to1^-} k(x) = 4 = k(1)$ and $\lim_{x\to1^+} k(x) = \frac{8}{A+1}$.
For continuity at $x = 1$ we therefore need $4 = \frac{8}{A+1}$ which gives $A = 1$.
For continuity at $x = 2$, we consider left and right limits at 2.
Note that $\lim_{x\to2^-} k(x) = \frac{8+10+1}{4A+1} = \frac{19}{4A+1}$ and $\lim_{x\to2^+} k(x) = \ln(2B) + 3 = k(2)$.
We found above that $A = 1$, so $\frac{19}{4A+1} = \frac{19}{5} = 3.8$.
For continuity at $x = 2$ we therefore need $\frac{19}{5} = \ln(2B) + 3$, giving $\ln(2B) = \frac{4}{5}$. By definition of the natural logarithm (or exponentiation), we find $2B = e^{4/5}$ and $B = \frac{1}{2}e^{4/5} = 0.5e^{0.8}$.

Answer: $A =$ 1 and $B = \frac{1}{2}e^{4/5} = 0.5e^{0.8}$

6. [5 points] Let

$$Q(v) = 1 + \arctan(v^{2v-1}).$$

Use the limit definition of the derivative to write an explicit expression for $Q'(4)$. *Your answer should not involve the letter Q. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $Q'(4) = \lim_{h \to 0} \frac{1 + \arctan\left((4+h)^{2(4+h)-1}\right) - \left(1 + \arctan(4^{2(4)-1})\right)}{h}$

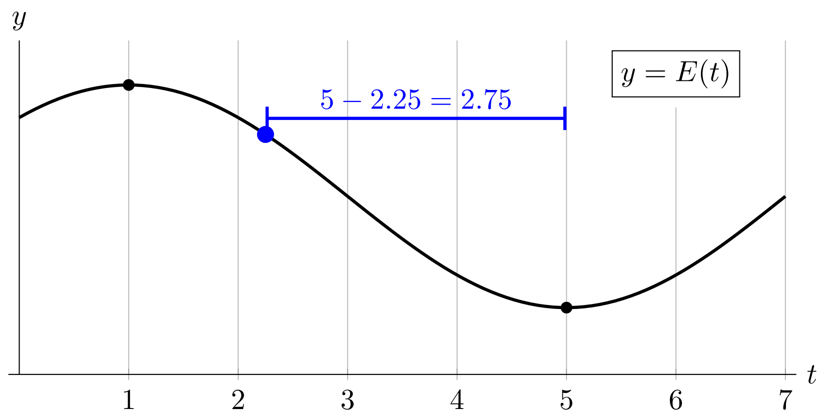
7. [7 points]

a. [4 points] Zoey, a zoologist, is studying the population of giraffes near a lake. She notices that the number of giraffes near the lake fluctuates in a sinusoidal manner over a 24 hour cycle. The giraffe population reaches a minimum of 30 giraffes at 7:00am every day, and rises to a maximum of 50 giraffes at 7:00pm every day. Let $G(t)$ be a sinusoidal function modeling the number of giraffes at the lake t hours after 6:00am.
Find a formula for $G(t)$.

Solution: Because we are given the times for maximum and minimum values, we choose to use transformations of the cosine function to model this situation (though using the sine function requires only a different horizontal shift).
Then $G(t)$ has the form $G(t) = A\cos(B(t-C)) + D$ for some constants A, B, C, and D.
The amplitude is $(50-30)/2 = 10$ so $|A| = 10$. The period is 24 so $B = \pi/12$. The midline is at $y = (50+30)/2 = 40$ so $D = 40$. Finally, a minimum occurs at $t = 1$ so we can use a vertical reflection (giving $-\cos$) combined with a horizontal shift to the right by 1. This would give $C = 1$ and $A = -10$. Therefore, one possible equation is $G(t) = -10\cos(\frac{\pi}{12}(t-1)) + 40$.

Answer: $G(t) =$ $-10\cos\left(\frac{\pi}{12}(t-1)\right) + 40$

b. [3 points] Zoey also studies the population of elephants in the area. Let $E(t)$ be a sinusoidal function modeling the number of elephants at the lake t hours after 6:00am. A portion of the graph of $E(t)$ is shown below.



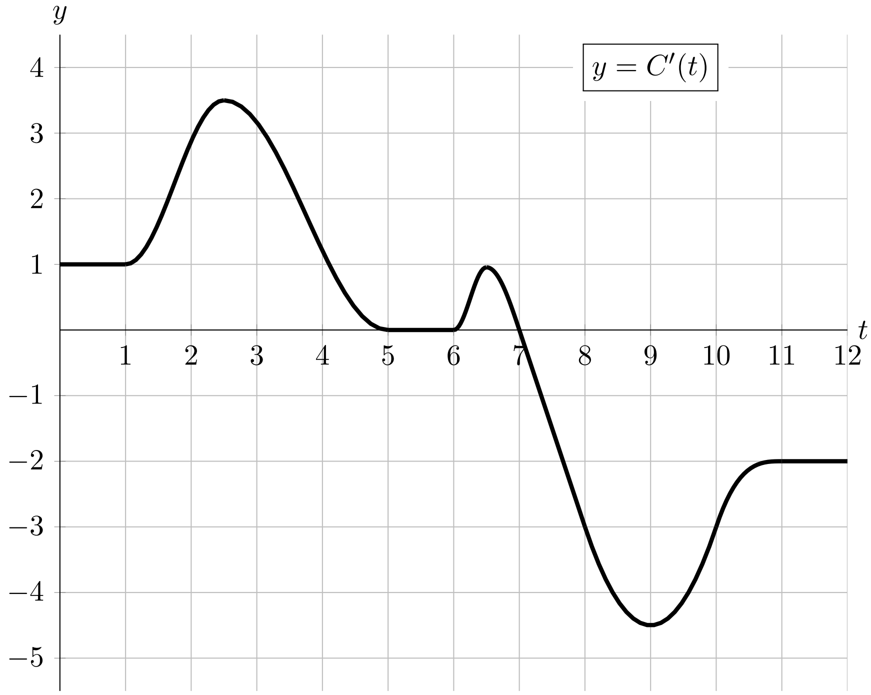
Give the **exact** values of the next two times t when this model predicts there will be the same number of elephants near the lake as there are at $t = 2.25$ (8:15am). You do not need to show work, but limited partial credit may be awarded for work shown.

Solution: To find the first time, we use symmetry of the graph. Because $5 - 2.25 = 2.75$, the next time there will be the same number of elephants near the lake as there are at $t = 2.25$ is at $t = 5 + 2.75 = 7.75$ hours after 6:00 am.
The second time occurs one full period after $t = 2.25$. It takes $5-1 = 4$ hours for the population of elephants to fall from a maximum to a minimum. Therefore, the period of $E(t)$ is 8 and so the second required time is given by $t = 2.25 + 8 = 10.25$ hours after 6:00am.

Answer: $t =$ 7.75

Answer: $t =$ 10.25

8. [6 points] Let $C(t)$ be the temperature, in degrees Fahrenheit (°F), of a cat café t hours after noon on a certain winter day. The function $C'(t)$, the **derivative** of $C(t)$, is graphed below.



a. [2 points] Over which of the following intervals of t, if any, is the temperature of the cat café constant? Circle **all** correct answers.

$[0, 1]$ $\quad$ $[5, 6]$ $\quad$ $[7, 8]$ $\quad$ $[11, 12]$ $\quad$ NONE OF THESE

b. [2 points] Over which of the following intervals of t, if any, is the temperature of the cat café decreasing? Circle **all** correct answers.

$[2, 3]$ $\quad$ $[3, 4]$ $\quad$ $[8, 9]$ $\quad$ $[9, 10]$ $\quad$ NONE OF THESE

c. [1 point] At which of the following times t is the temperature in the cat café changing most rapidly? Circle the **one** correct answer.

$t = 1.5$ $\quad$ $t = 2.5$ $\quad$ $t = 8$ $\quad$ $t = 9$

d. [1 point] At which of the following times t is the temperature in the cat café the highest? Circle the **one** correct answer.

$t = 0$ $\quad$ $t = 2.5$ $\quad$ $t = 7$ $\quad$ $t = 12$