Math 115 — First Midterm — February 7, 2023

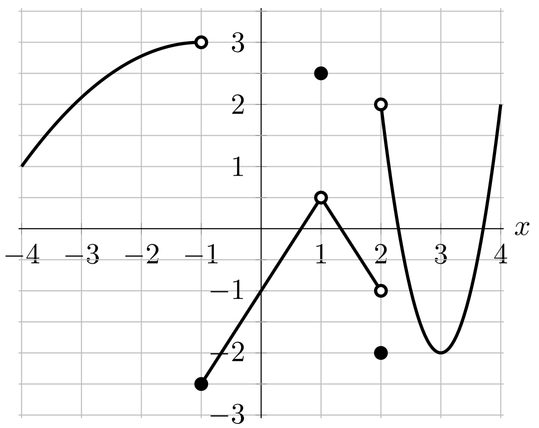
EXAM SOLUTIONS

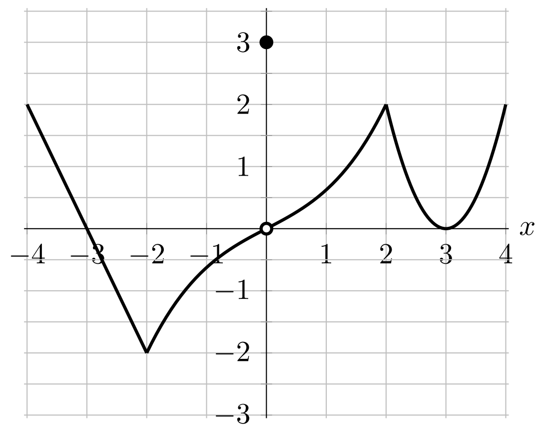
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 8 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded. No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	10	
2	8	
3	5	
4	5	

Problem	Points	Score
5	10	
6	10	
7	8	
8	4	
Total	60	

1. [10 points] Given below are the graphs of two functions, $P(x)$ and $Q(x)$, both defined on the interval $(-4, 4)$. Use these graphs to answer the questions below about $P(x)$ and $Q(x)$. *You do not need to show work.*





a. [1 point] Circle all of the x values below at which the function $P(x)$ is *not* continuous.

$x = -3$ $\quad$ $\boxed{x = -1}$ $\quad$ $\boxed{x = 1}$ $\quad$ $x = 3$ $\quad$ NONE OF THESE

b. [6 points] Find the **exact** numerical value of each expression below, if possible. For any values that do not exist, including if they are limits that diverge to $\pm\infty$, write DNE.

i. $\lim_{x \to 1} P(x) =$ 1/2

ii. $\lim_{x \to 2^-} P(x) =$ -1

iii. $\lim_{x \to Q(0)} P(x) =$ -2

iv. $\lim_{x \to 1} Q(3x) =$ 0

v. $\lim_{x \to 2} \big(P(x)Q(x)\big) =$ **DNE**

vi. $\lim_{x \to 0} \dfrac{P(x) - P(0)}{x} =$ 3/2

c. [2 points] Circle all x-values given below where the function $\dfrac{1}{Q(x)}$ has a vertical asymptote.

$\boxed{x = -3}$ $\quad$ $x = -2$ $\quad$ $\boxed{x = 0}$ $\quad$ $x = 2$ $\quad$ $\boxed{x = 3}$ $\quad$ NONE OF THESE

d. [1 point] Circle all the x-values given below where the function $\dfrac{1}{Q(x)}$ is undefined.

$\boxed{x = -3}$ $\quad$ $x = -2$ $\quad$ $x = 0$ $\quad$ $x = 2$ $\quad$ $\boxed{x = 3}$ $\quad$ NONE OF THESE

2. [8 points] Octavia wants to inflate her front bicycle tire with her new air compressor, so she attaches the compressor to her tire and turns it on. Let $V(t)$ represent the total volume of air, in cubic inches, that the compressor has pumped into the tire t seconds after the compressor has been switched on, and let $P(s)$ be the air pressure, in pounds per square inch (psi), inside the tire when it has been filled with s cubic inches of air. Assume both functions are invertible and differentiable.

a. [2 points] Write a complete sentence that gives a practical interpretation of the equation

$$P^{-1}(90) = 200.$$

Solution: When the tire has been filled with 200 cubic inches of air, the pressure inside of it is 90 psi.

b. [2 points] Write a complete sentence that gives a practical interpretation of the equation

$$P(V(30)) = 90.$$

Solution: Thirty seconds after Octavia switched on the air compressor, the pressure inside the tire is 90 psi.

c. [1 point] Assuming the equations in parts (a) and (b) are true, how many cubic inches of air has the compressor pumped into the tire after it has been running for 30 seconds?

Answer: 200

d. [3 points] Circle **all** valid practical interpretations of the equation

$$V'(20) = 10.$$

i. In the first 20 seconds it is running, the air compressor pumps about 10 cubic inches of air.

ii. Twenty seconds after the air compressor started running, if Octavia runs it for another 10 seconds, the total volume it has pumped will increase by about 1 cubic inch per second.

iii. In the twentieth second after Octavia turns on her air compressor, the total volume of air that it has pumped increases by about 10 cubic inches.

iv. When 20 cubic inches of air have been pumped, the air compressor is pumping air at a rate of about 10 cubic inches per second.

v. After the air compressor has been running for 20 seconds, the compressor will pump about 10 cubic inches of air in the next second.

3. [5 points] Suppose $f(t)$ is a differentiable function whose tangent line at the point $t = 1$ is given by the linear function $L(t)$. To the right is a table consisting of some values of $f(t)$ and $L(t)$.

t	-1	1	3	5
$f(t)$	5	2	2	9
$L(t)$	3	2	1	0

a. [1 point] Find the average rate of change of $f(t)$ on the interval $[-1, 5]$.

Solution: $\dfrac{f(5) - f(-1)}{5 - (-1)} = \dfrac{9-5}{5-(-1)} = \dfrac{4}{6} = \dfrac{2}{3}$

Answer: 2/3

b. [1 point] Find the instantaneous rate of change of $f(t)$ at $t = 1$.

Solution: This is equal to the slope of the tangent line $L(t)$ at $t = 1$, which is $\dfrac{L(1) - L(-1)}{1 - (-1)} = \dfrac{2-3}{2} = \dfrac{-1}{2}$

Answer: $-1/2$

c. [1 point] Using the table, find the best possible estimate of $f'(-1)$.

Solution: $\dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{2-5}{1-(-1)} = \dfrac{-3}{2}$

Answer: $-3/2$

d. [2 points] The function L is invertible, and its inverse function L^{-1} is also linear. Find numbers m and b such that $L^{-1}(x) = mx + b$.

Solution: We have $L(t) = -\frac{1}{2}t + \frac{5}{2}$. So we solve the equation $x = -\frac{1}{2}y + \frac{5}{2}$ for y, which gives $y = -2x + 5$.

Answer: $m =$ -2 and $b =$ 5

4. [5 points] Let $f(x)$, $g(x)$, and $h(x)$ be the functions defined for all real numbers by

$$f(x) = 2^{c+1}c^x, \qquad g(x) = e^c\cos(cx), \quad \text{and} \quad h(x) = \begin{cases} f(x) & x \le 0 \\ g(x) & x > 0 \end{cases}$$

where c is a nonzero constant. In each part below, find an exact value for the constant c so that the given condition holds. (*Your value for the constant c may be different in each part.*)

a. [1 point] The function $f(x)$ has a continuous decay rate of 15%.

Solution: If we write an exponential function in the form $P = P_0e^{-kx}$ with $k > 0$, then k is its continuous decay rate. So in order for $2^{c+1}c^x = 2^{c+1}e^{(\ln c)x}$ to have a continuous decay rate of 15%, we want $\ln c = -.15$, so $c = e^{-.15}$.

Answer: $e^{-0.15}$

b. [1 point] The function $g(x)$ has a period of 3.

Solution: The cosine function has a period of 2π, so $g(x)$ has a period of $\frac{2\pi}{c}$. So we solve $3 = \frac{2\pi}{c}$ to get $c = \frac{2\pi}{3}$.

Answer: $\dfrac{2\pi}{3}$

c. [3 points] The function $h(x)$ is continuous at zero.

Solution: Since $f(x)$ and $g(x)$ are both continuous, in order for $h(x)$ to be continuous at zero we need $f(0) = g(0)$. Since $f(0) = 2^{c+1}c^0 = 2^{c+1}$ and $g(0) = e^c\cos(0) = e^c$, this means we need $2^{c+1} = e^c$. Solving this for c gives

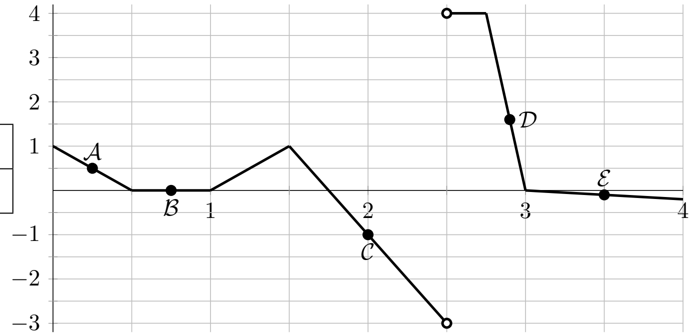
$$c \;=\; \ln e^c \;=\; \ln(2^{c+1}) \;=\; (c+1)\ln 2 \;=\; c\ln 2 + \ln 2.$$

Thus $c(1 - \ln 2) = c - c\ln 2 = \ln 2$, so $c = \dfrac{\ln 2}{1 - \ln 2}$.

Answer: $\dfrac{1 - \ln 2}{\ln 2}$

5. [10 points] We tracked the Cheshire Kid far out in the desert as he attempted to find buried gold. After checking his map at time $t = 0$, the Cheshire Kid moved *only* in the north/south direction, with the function $D(t)$ giving his position in kilometers north of his starting point t hours after he checked his map at time $t = 0$. We now consider the following graph of the **derivative** $D'(t)$, along with a table to the left that shows a few values of $D(t)$.

t	1	1.5	1.75	2
$D(t)$	0.25	0.5	0.625	0.5



a. [2 points] At time $t = 2$, is the Cheshire Kid *north* or *south* of his position at $t = 1$? By how much? Give your answer by circling NORTH or SOUTH and filling in the appropriate number of kilometers in the sentence below:

Solution: At $t = 2$, he is 0.25 kilometers (NORTH / SOUTH) of his position at $t = 1$.

b. [2 points] Find all times t for $0 < t < 4$ when the Cheshire Kid is traveling at his maximum *speed*. Give your answer as value(s) and/or interval(s) of t.

Solution: $2.5 < x \leq 2.75$ or $(2.5, 2.75]$

c. [2 points] Rank the points $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ (as shown on the graph) in order of *descending velocity*, i.e., starting with the point where the Cheshire Kid's velocity is *greatest* and ending with the point where it is *least*.

Solution: $\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$, $\mathcal{C}$

d. [2 points] Find the average velocity of the Cheshire Kid over the time interval $[1, 2]$. *Include units.*

Solution: $\frac{D(2)-D(1)}{2-1} = \frac{0.5-0.25}{2-1} = 0.25$ km/hr

e. [2 points] Circle all of the following intervals over which the Cheshire Kid is always traveling *south*:

$(0, 0.5)$ $(1.5, 2)$ $(2, 2.5)$ $(2.5, 3)$ $(3.5, 4)$ NONE OF THESE

6. [10 points] Let $\ell(t) = \dfrac{1}{1+e^{-t}}$.

a. [5 points] Use the limit definition of the derivative to write an explicit expression for $\ell'(-2)$. *Your answer should not involve the name of the function, ℓ. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $\ell'(-2) =$ $\displaystyle\lim_{h\to 0} \frac{\frac{1}{1+e^{-(-2+h)}} - \frac{1}{1+e^{2}}}{h}$ or $\displaystyle\lim_{x\to -2} \frac{\frac{1}{1+e^{-x}} - \frac{1}{1+e^{2}}}{x+2}$

b. [3 points] Evaluate each of the limits below. Give exact answers. If a limit does not exist, including if it diverges to $\pm\infty$, write DNE. You do not need to show work.

i. $\displaystyle\lim_{t\to 0} \ell(t)$

Answer: 1/2

ii. $\displaystyle\lim_{t\to \infty} \ell(t)$

Answer: 1

iii. $\displaystyle\lim_{t\to -\infty} \ell(t)$

Answer: 0

c. [2 points] The function $\ell(t)$ defined above is called the *logistic* function, and you encountered it on Team HW 2. The logistic function is neither even nor odd, but it is possible to shift the graph of $\ell(t)$ to obtain a function that *is* even or odd. Fill in the blanks below to make a TRUE STATEMENT, following these guidelines:

- in the first blank, write UP, DOWN, LEFT, or RIGHT;
- in the second blank, write a positive number;
- in the third blank, write EVEN or ODD.

No justification is needed.

Solution: Any shift (vertical or horizontal) of an even function will have the same limit at $-\infty$ that it has at ∞, so by i. and iii. of part (b) above we know the function we are looking for must be *odd*. And since $\ell(0) = \frac{1}{2}$ and the graph of any odd function must pass through the origin, we need to shift $\ell(t)$ *down* by $\frac{1}{2}$.

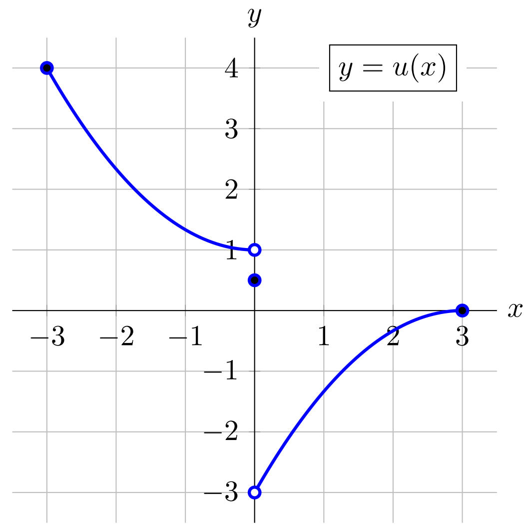
"If you shift the graph of $\ell(t)$ DOWN by $\frac{1}{2}$, the resulting function is ODD."

7. [8 points] For each part below, carefully draw the graph of a single function on the given axes that satisfies the given conditions.

a. [4 points]

A function $u(x)$, defined for all $-3 \leq x \leq 3$, that satisfies all of the following:

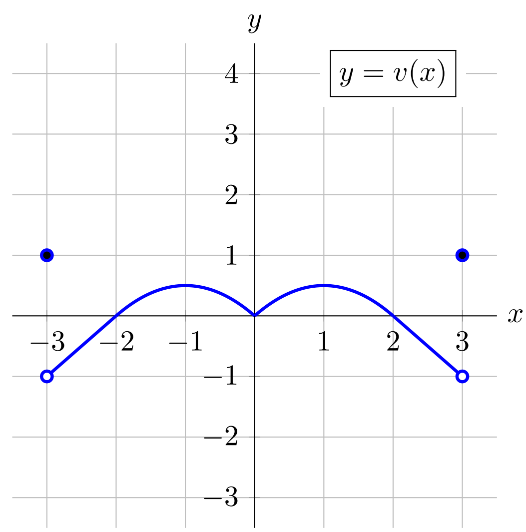
- $u(x)$ is invertible;
- $u(x)$ is decreasing and concave up on $(-3, 0)$;
- $u(x)$ is increasing and concave down on $(0, 3)$;
- $u(x)$ is *not* continuous at $x = 0$, but *is* continuous on the intervals $(-3, 0)$ and $(0, 3)$.



b. [4 points]

A function $v(x)$, defined for all $-3 \leq x \leq 3$, that satisfies all of the following:

- $v(x)$ is an even function;
- $v'(2) = -1$
- $\lim_{x \to 3^-} v(x)$ exists but does not equal $v(3)$.
- $\lim_{h \to 0^+} \dfrac{v(0+h) - v(0)}{h} = 1$



8. [4 points] Recall from Team HW 2 that if the function $f(x)$ is not defined at a, we say that $f(x)$ can be *continuously extended* to a if there is a number c such that the piecewise defined function

$$F(x) = \begin{cases} f(x) & x \neq a \\ c & x = a \end{cases}$$

is continuous at a. Write down a formula for a rational function $r(x)$ that satisfies all of the following conditions, or, if no such rational function exists, write DNE:

- the domain of $r(x)$ is all real numbers except for 0 and 3;
- $r(x)$ can be continuously extended to 0;
- $r(x)$ *cannot* be continuously extended to 3.

Solution: The simplest solution is $r(x) = \dfrac{x}{x(x-3)}$, although many other rational functions will work too. The important thing is that both x and $x - 3$ appear as factors in the denominator, the exponent on x is at least as big in the numerator as it is in the denominator, and the exponent on $(x - 3)$ is bigger in the denominator than in the numerator.

Answer: $r(x) =$ $\dfrac{x}{x(x-3)}$