Math 115 — Second Midterm — March 21, 2023

EXAM SOLUTIONS

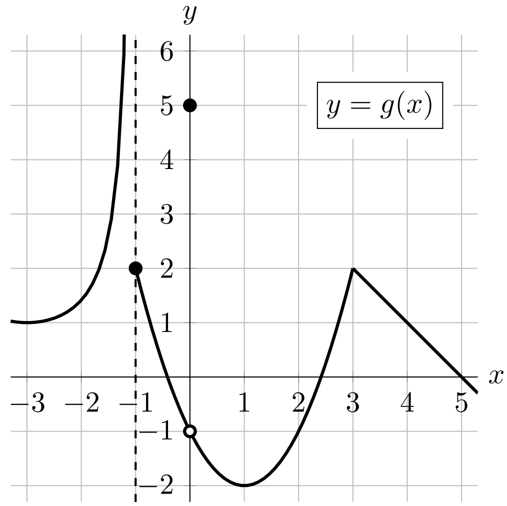
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded. No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	10	
2	8	
3	9	
4	10	
5	4	

Problem	Points	Score
6	7	
7	8	
8	6	
9	6	
10	12	
Total	80	

1. [10 points]
A portion of the graph of a function $g(x)$ is shown to the right, along with some values of an invertible, differentiable function $h(x)$ and its derivative $h'(x)$ below. Note that:

- $g(x)$ is linear on $[3, 5]$;
- $g(x)$ has a vertical asymptote at $x = -1$.



x	-2	0	2	4	6
$h(x)$	-1	$-e^{-1}$	0	$\sqrt{2}$	e
$h'(x)$	2	1	π	5	$\sqrt{3}$

a. [2 points] Let $M(x) = x^2 h(x)$. Find $M'(-2)$.

Solution: By the Product Rule, $M'(x) = 2xh(x) + x^2 h'(x)$, so

$$M'(-2) = -4h(-2) + 4h'(-2) = (-4)(-1) + 4(2) = 12.$$

Answer: $M'(-2) =$ 12

b. [2 points] Let $K(x) = \dfrac{g(x)}{h(x)}$. Find $K'(4)$.

Solution: Using the Quotient Rule,

$$K'(4) = \frac{g'(4)h(4) - g(4)h'(4)}{h(4)^2} = \frac{(-1)(\sqrt{2}) - (1)(5)}{2} = \frac{-\sqrt{2} - 5}{2}.$$

Answer: $K'(4) = \dfrac{-\sqrt{2} - 5}{2}$

c. [2 points] Find $(h^{-1})'(0)$.

Solution: Using the rule for the derivative of an inverse function,

$$(h^{-1})'(0) = \frac{1}{h'\big(h^{-1}(0)\big)} = \frac{1}{h'(2)} = \frac{1}{\pi}.$$

Answer: $(h^{-1})'(0) = \dfrac{1}{\pi}$

d. [2 points] On which of the following intervals does $g(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers.

$[-3, -1]$ $[0, 2]$ **[1, 3]** $[2, 4]$ NONE OF THESE

e. [2 points] On which of the following intervals does $g(x)$ satisfy the conclusion of the Mean Value Theorem? Circle all correct answers.

[−3, −1] $[0, 2]$ **[1, 3]** $[2, 5]$ NONE OF THESE

2. [8 points] Throughout this problem, let $K(x) = e^x - ex$. In case it is helpful, $e \approx 2.7$.

a. [1 point] Find a formula for $K'(x)$.

Answer: $K'(x) =$ $e^x - e$

b. [4 points] Find the x-coordinate of all global minimum(s) and global maximum(s) of $K(x)$ **on the interval** $[0, 3]$. If there are none of a particular type, write NONE. Use calculus to find your answers, and make sure that you show enough evidence to justify your conclusions.

Solution: Since $K'(x) = e^x - e$, we have that $K'(x) = 0$ exactly when $x = 1$, which is within our domain. Since $K(x)$ is continuous on $[0, 3]$, it must have a global max and min on $[0, 3]$ by the Extreme Value Theorem, and these must occur at the critical point $x = 1$ or at an endpoint of $[0, 3]$. Since $K(0) = 1$, $K(1) = 0$, and $K(3) = e^3 - 3e = e(e^2 - 3) > 2(2^2 - 3) = 2$, we see that $K(x)$ has a min at $x = 1$ and a max at $x = 3$ on the interval $[0, 3]$.

Answer: Global min(s) at $x =$ **1**

Answer: Global max(es) at $x =$ **3**

c. [2 points] Find the linear approximation $L(x)$ of the function $K(x)$ at the point $x = 0$.

Solution: The linear approximation $L(x)$ of the function $K(x)$ at the point $x = 0$ is

$$L(x) = K(0) + K'(0)(x - 0) = 1 + (1 - e)x.$$

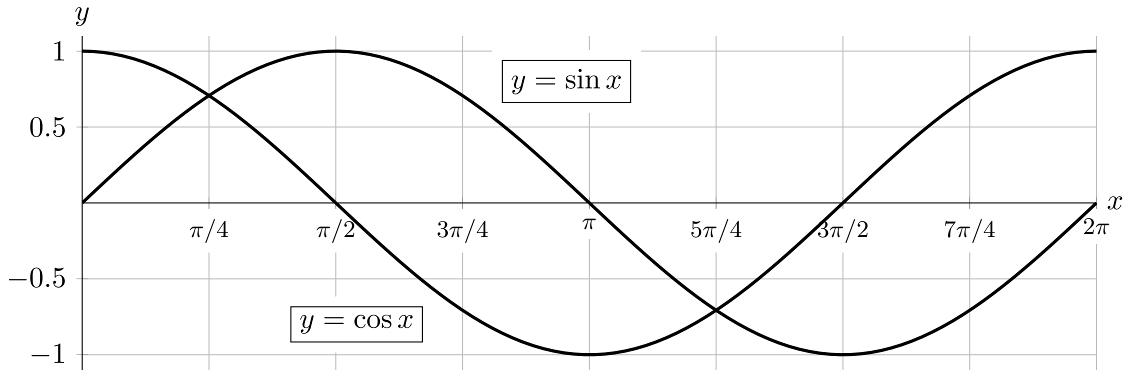
Answer: $L(x) =$ $1 + (1 - e)x$

d. [1 point] If you were to use the linear approximation that you found in part c. to estimate $K(0.1)$, would the approximation give you an *underestimate* or *overestimate* of the true value of $K(0.1)$? Circle the correct answer, or circle NEI if there is not enough information to decide.

[UNDERESTIMATE] OVERESTIMATE NEI

Solution: Since $K''(x) = e^x > 0$ for all x, the function $K(x)$ is concave up everywhere, so any linear approximation to $K(x)$ will always give underestimates.

3. [9 points] Throughout this problem, let $f(x) = \sin x + \cos x$. For reference, you may use the graphs of sine and cosine given below, but note that neither of these is a graph of f, since f is their *sum*.



a. [1 point] Give a formula for the derivative of $f(x)$.

Answer: $f'(x) =$ $\cos x - \sin x$

b. [2 points] On which of the following intervals is $f(x)$ increasing? Circle all correct answers.

$\boxed{\left(0, \frac{\pi}{4}\right)}$ $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ $\boxed{\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)}$ $\boxed{\left(\frac{7\pi}{4}, 2\pi\right)}$ NONE OF THESE

c. [2 points] On which of the following intervals is $f(x)$ concave down? Circle all correct answers.

$\boxed{\left(0, \frac{\pi}{4}\right)}$ $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ $\boxed{\left(\frac{7\pi}{4}, 2\pi\right)}$ NONE OF THESE

d. [4 points] Find and classify all local extrema of $f(x)$ on the interval $(0, 2\pi)$. If there are none of a particular type, write NONE. Use calculus to find and justify your answers, and show all your work.

Solution: Since $f(x)$ is differentiable and $(0, 2\pi)$ does not contains its endpoints, any local extremum of $f(x)$ on $(0, 2\pi)$ must occur at a point where $f'(x) = 0$, that is, where $\cos(x) = \sin(x)$. From the graph we see that this happens only at $x = \pi/4$ and $x = 5\pi/4$. Since $f''(x) = -\sin x - \cos x = -f(x)$, from the graph we see that $f''\left(\frac{\pi}{4}\right) < 0$ and $f''\left(\frac{5\pi}{4}\right) > 0$, so $\frac{\pi}{4}$ is a local max and $\frac{5\pi}{4}$ is a local min by the Second Derivative Test.

Answer: Local min(s) at $x =$ $\frac{5\pi}{4}$ and Local max(es) at $x =$ $\frac{\pi}{4}$

4. [10 points] Octavia is now inflating her rear bicycle tire with her new air compressor. Let $V(t)$ be the total volume of air, in cubic inches, that the compressor has pumped into the tire t seconds after the compressor has been switched on, and let $P(s)$ be the air pressure, in pounds per square inch (psi), inside the tire when it has been filled with s cubic inches of air.

Octavia has learned that increasing the pressure in her tire will reduce the *rolling resistance* of the tire, which is the energy that is lost, per unit of time, while the tire is rolling. Let $R(p)$ be the rolling resistance, measured in watts, of the tire when the air pressure inside the tire is p psi. Assume the functions V, P, and R are invertible and differentiable.

a. [2 points] Write a number in the blank below to give a practical interpretation of the equation

$$(R^{-1})'(43) = -19.$$

If Octavia wants to reduce the rolling resistance of her tire from 43 to 41 watts, she should increase her tire pressure by about 38 *psi.*

b. [2 points] Circle the **one** equation below that best represents the statement: "If Octavia wants the rolling resistance of her rear wheel to be 32 watts, she needs to run the air compressor for 30 seconds."

(i) $R^{-1}\big(P^{-1}(V^{-1}(32))\big) > 30$ **[(ii)]** $P\big(V(30)\big) = R^{-1}(32)$

(iii) $R'\big(P'(V'(30))\big) = 32$ (iv) $R(32) < P\big(V(30)\big)$

c. [2 points] Write a mathematical equation involving a derivative that has the following practical interpretation: "If Octavia increases her tire pressure from 50 psi to 60 psi, she will reduce the tire's rolling resistance by about a half a watt.

Answer: $R'(50) = -\frac{1}{20}$

d. [2 points] Octavia knows that as she increases her tire pressure, the corresponding reduction in rolling resistance decreases as the tire inflates. Therefore the graph of $R(p)$ is:

(i) increasing and concave up   (ii) increasing and concave down

[(iii)] decreasing and concave up   (iv) decreasing and concave down

e. [2 points] After experimenting and doing some research, Octavia concludes that V and P are closely modeled by the equations

$$V(t) = \frac{20}{3}t \qquad \text{and} \qquad P(s) = \frac{3}{10}s + 30,$$

and for values of p between 40 and 100 psi, $R'(p)$ is about -0.05. Given this, estimate the rate at which Octavia is reducing her rolling resistance by inflating her tire, in watts per second, when the compressor has been running for 30 seconds. *Show your work.*

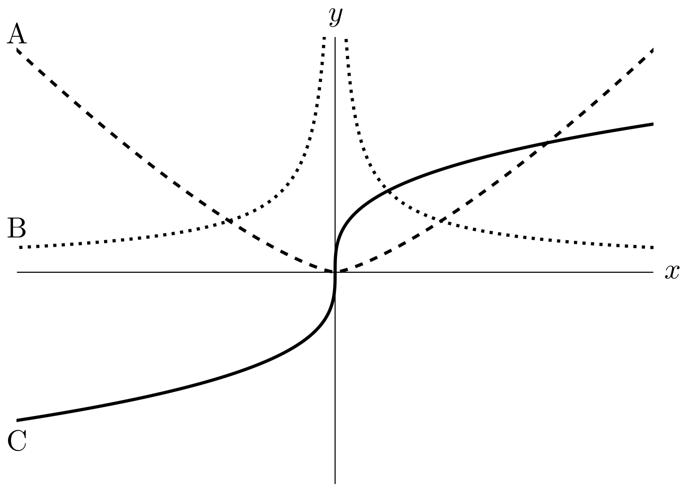
Solution: The rate at which Octavia is reducing her rolling resistance is $\frac{dR}{dt}$. Writing $s = V(t)$ and $p = P(s)$ and applying the Chain Rule gives us

$$\frac{dR}{dt} = \frac{dR}{dp} \cdot \frac{dp}{ds} \cdot \frac{ds}{dt} = R'(p) \cdot \frac{3}{10} \cdot \frac{20}{3} = 2R'(p).$$

Therefore, from the estimate $R'(p) \approx -0.05$, we conclude $\frac{dR}{dt}\big|_{t=30} = -0.1$ watts per second.

Answer: 0.1 **watts per second**

5. [4 points] Shown below are portions of the graphs of $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Note that the dotted graph has a vertical asymptote at $x = 0$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.



Answer: $f(x)$: A

$f'(x)$: C

$f''(x)$: B

6. [7 points] The function $q(x)$ is given by the following formula, where c and m are constants:

$$q(x) = \begin{cases} c - 4x - x^2 & -3 \le x \le 0 \\ mx & 0 < x \le 2. \end{cases}$$

a. [4 points] Assuming $c = -3$ and $m = 2$, find the x-values of all global minima and global maxima of $q(x)$ on the interval $[-3, 2]$. If there are none of a particular type, write NONE. Use calculus to find and justify your answers, and show your work.

Solution: On $(-3, 0)$ we have $q'(x) = -4 - 2x$, so the only critical point of $q(x)$ on $(-3, 0)$ is $x = -2$. Since q is continuous on $[-3, 0]$ and we have $q(-3) = 0$, $q(-2) = 1$, and $q(0) = -3$, we see that, on the interval $[-3, 0]$, $q(x)$ has a max of 1 at -2 and a min of -3 at 0.

Now we check $(0, 2]$. On $(0, 2]$, $q(x)$ is linear with slope 2, so the max value of $q(x)$ on $(0, 2]$ is $q(2) = 4$, and we have $q(x) > \lim_{x \to 0^+} q(x) = 0$ for all $0 < x \le 2$.

Putting this all together, we find that on the interval $[-3, 2]$, the function $q(x)$ has a global maximum of 4 at $x = 2$, and a global minimum of -3 at $x = 0$.

Answer: Global min(s) at $x =$ 0 and Global max(es) at $x =$ 2

b. [3 points] Find one pair of values for c and m such that $q(x)$ is differentiable at $x = 0$. Show your work.

Solution: To be differentiable at $x = 0$, the function $q(x)$ must be continuous at 0, which means

$$c - 4 \cdot 0 - 0^2 = m \cdot 0,$$

that is, $c = 0$. But we will also need the derivative of $c - 4x - x^2$ at zero to equal the derivative of mx at zero. Thus $-4 - 2 \cdot 0 = m$, so $m = -4$.

Answer: $c =$ 0 and $m =$ -4

7. [8 points] Allie, Minnie, and Ian have decided to start a business producing soda cans. They've received their first order of 100,000 cans, but haven't settled on a final product design yet. They want their cylindrical cans to hold 355 mL of liquid, so if the cans have base radius r and height h, both in centimeters, then by the formula for the volume of a cylinder,

$$\pi r^2 h = 355.$$

Given the cost of aluminum, Allie, Minnie, and Ian calculate that producing 100,000 cans with a base radius r and height h would cost

$$k\pi r(r+h)$$

dollars, where $k = 12.96$.

a. [2 points] For a given radius r, let $C(r)$ equal the cost of producing 100,000 aluminum cans with radius r and a volume of 355 mL. Find a formula for $C(r)$. Your formula should not include h, but can include k.

Answer: $C(r) = k\pi r\left(r + \frac{355}{\pi r^2}\right) = k\pi r^2 + \frac{355k}{r}$

b. [1 point] What is the domain of $C(r)$?

Answer: $(0, \infty)$

c. [5 points] Find the radius r that minimizes the cost of producing 100,000 soda cans. Use calculus to find and justify your answer, and be sure to show enough evidence that the value(s) you find do in fact minimize the cost.

Solution: Differentiating $C(r)$ gives us $C'(r) = 2k\pi r - 355k/r^2$, which is defined everywhere on $(0, \infty)$ and equals 0 exactly when

$$2k\pi r = \frac{355k}{r^2}, \quad \text{that is, when} \quad r = \sqrt[3]{\frac{355}{2\pi}}.$$

This is the only critical point of $C(r)$ on $(0, \infty)$. Since the domain of $C(r)$ does not include any endpoints, if $C(r)$ has a minimum (or a maximum) then it must be at this point. Checking the one-sided limits of $C(r)$ at the endpoints of its domain gives

$$\lim_{r\to 0^+} C(r) = \infty = \lim_{r\to\infty} C(r),$$

so the critical point $\sqrt[3]{\frac{355}{2\pi}}$ must be the global minimum of $C(r)$ on $(0, \infty)$, and is therefore the radius that minimizes cost.

Answer: value(s) of r that minimize the cost: $\sqrt[3]{\frac{355}{2\pi}}$

8. [6 points] The equation $x^2 + xy + 2y^2 = 28$ defines y implicitly as a function of x.

a. [4 points] Compute $\frac{dy}{dx}$. Show every step of your work.

Solution: Implicitly differentiating the equation $x^2 + xy + 2y^2 = 28$ with respect to x gives:

$$2x + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0.$$

Now collect like terms, and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(x + 4y\right) = -2x - y, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{-2x - y}{x + 4y}.$$

Answer: $\frac{dy}{dx} = \frac{-2x - y}{x + 4y}$

b. [2 points] Find an equation of the line tangent to the curve defined by $x^2 + xy + 2y^2 = 28$ at the point $(2, 3)$.

Solution: Using point-slope form, the tangent line will have equation $y - 3 = m(x - 2)$, where m is the slope. And, using part (a), the slope of the tangent line at $(2, 3)$ is

$$m = \frac{dy}{dx}\Big|_{(x,y)=(2,3)} = \frac{-2(2) - 3}{2 + 4(3)} = \frac{-7}{14} = -\frac{1}{2}.$$

So the line has equation $y - 3 = -\frac{1}{2}(x - 2)$.

Answer: $y - 3 = -\frac{1}{2}(x - 2)$

9. [6 points] The equation $x + \frac{1}{3}y^3 - y = 1$ implicitly defines x and y as functions of each other. Implicitly differentiating this equation with respect to x and solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{-1}{y^2 - 1}.$$

Let $\mathcal{C}$ be the graph of the equation $x + \frac{1}{3}y^3 - y = 1$. *Note that all points listed as possible answers below <u>do</u> actually lie on the graph $\mathcal{C}$.*

a. [2 points] Circle all points below at which the line tangent to $\mathcal{C}$ is *horizontal.*

$(-5, 3)$ $\quad$ $\left(\frac{1}{3}, -1\right)$ $\quad$ $(1, 0)$ $\quad$ $\left(1 + \frac{\sqrt{2}}{3}, \sqrt{2}\right)$ $\quad$ $\left(\frac{5}{3}, 1\right)$ $\quad$ [NONE OF THESE]

b. [2 points] Circle all points below at which the line tangent to $\mathcal{C}$ is *vertical.*

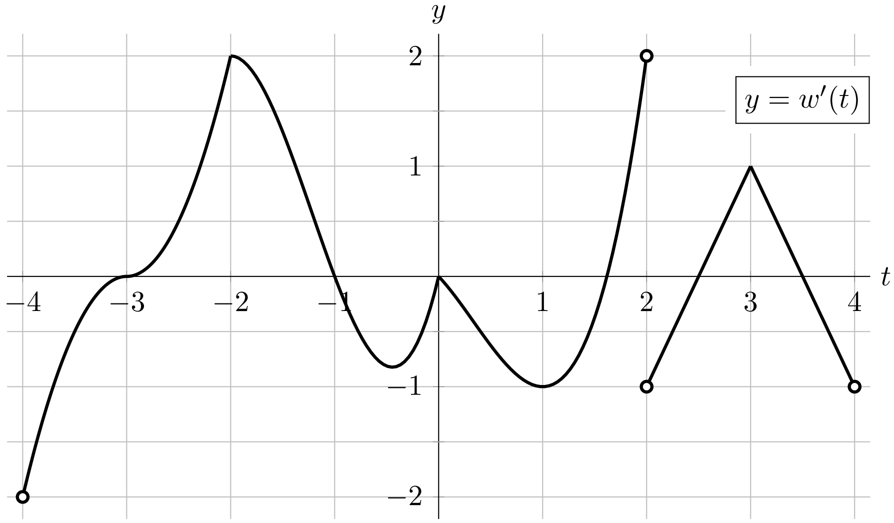
$(-5, 3)$ $\quad$ [$\left(\frac{1}{3}, -1\right)$] $\quad$ $(1, 0)$ $\quad$ $\left(1 + \frac{\sqrt{2}}{3}, \sqrt{2}\right)$ $\quad$ [$\left(\frac{5}{3}, 1\right)$] $\quad$ NONE OF THESE

c. [2 points] Circle all points below at which $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are equal to each other.

$(-5, 3)$ $\quad$ $\left(\frac{1}{3}, -1\right)$ $\quad$ [$(1, 0)$] $\quad$ [$\left(1 + \frac{\sqrt{2}}{3}, \sqrt{2}\right)$] $\quad$ $\left(\frac{5}{3}, 1\right)$ $\quad$ NONE OF THESE

Solution: $\frac{dy}{dx} = \frac{dx}{dy}$ when $\frac{-1}{y^2-1} = \frac{y^2-1}{-1}$, which happens when $(y^2 - 1)^2 = 1$. Thus $y^2 = 2$ or $y^2 = 0$, which means $y = 0$ or $y = \pm\sqrt{2}$.

10. [12 points] Suppose $w(t)$ is a continuous function, defined on the interval $(-4, 4)$. A graph of the **derivative** $w'(t)$ is given below.



a. [2 points] Circle all points below that are critical points of $w(t)$.

$\boxed{t=-3}$ $t=-2$ $t=1$ $\boxed{t=2}$ $t=3$ NONE OF THESE

b. [2 points] Circle all points below that are critical points of $w'(t)$.

$\boxed{t=-3}$ $\boxed{t=-2}$ $\boxed{t=1}$ $t=2$ $\boxed{t=3}$ NONE OF THESE

c. [2 points] Circle all points below that are local minima of $w(t)$.

$\boxed{t=-3}$ $t=-2$ $t=-1$ $t=1$ $t=2$ NONE OF THESE

d. [2 points] Circle all points below that are local maxima of $w(t)$.

$t=-3$ $t=-2$ $\boxed{t=-1}$ $t=1$ $\boxed{t=2}$ NONE OF THESE

e. [2 points] Circle all points below that are inflection points of $w(t)$.

$t=-3$ $\boxed{t=-2}$ $\boxed{t=1}$ $t=2$ $\boxed{t=3}$ NONE OF THESE

f. [1 point] Circle all points below that are global maxima of $w'(t)$ on the interval $(-4, 4)$.

$t=-4$ $\boxed{t=-2}$ $t=1$ $t=2$ $t=3$ NONE OF THESE

g. [1 point] Circle all points below that are global minima of $w'(t)$ on the interval $(-4, 4)$.

$t=-4$ $t=-2$ $t=1$ $t=2$ $t=3$ $\boxed{\text{NONE OF THESE}}$