Math 115 — First Midterm — February 13, 2024

EXAM SOLUTIONS

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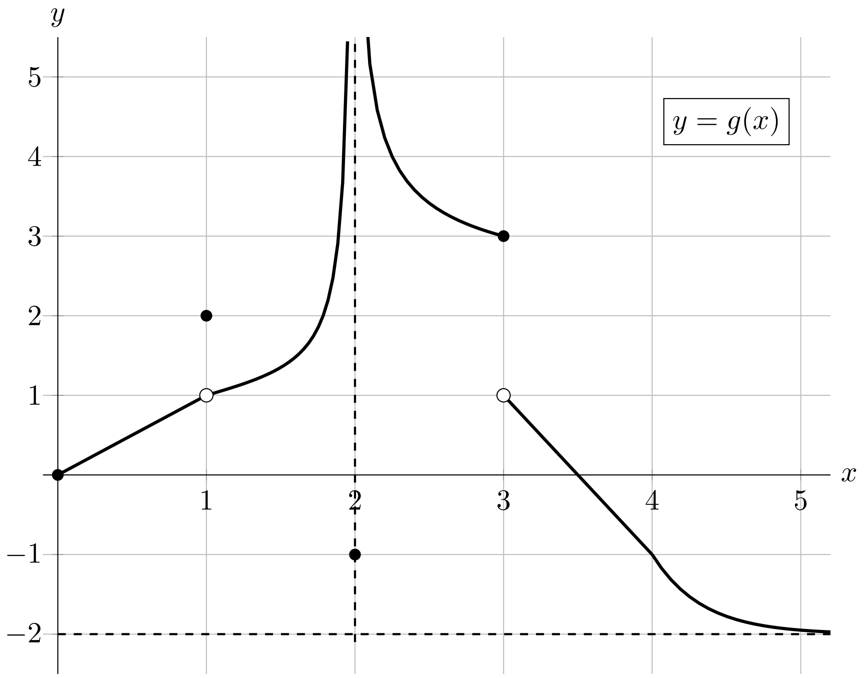
1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 8 pages including this cover.
3. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	8	
3	5	
4	6	
5	9	

Problem	Points	Score
6	9	
7	7	
8	7	
Total	60	

1. [9 points] Below is a portion of the graph of an odd function $g(x)$, which has domain $(-\infty, \infty)$ even though the graph below only shows part of the function with $x \geq 0$. Note that $g(x)$ is linear on the intervals $(0, 1)$ and $(3, 4)$, has a sharp corner at $x = 4$, has a vertical asymptote at $x = 2$, a horizontal asymptote at $y = -2$, and is decreasing for $x > 4$.



a. [1 point] At which of the following values of x is $g(x)$ continuous? *Circle all correct answers.*

$x = 1$ $x = 2$ $x = 3$ [✓ $x = 4$] NONE OF THESE

b. [8 points] Find the **exact** numerical value of each expression below, if possible. For any values that do not exist, including if they are limits that diverge to $\pm\infty$, write DNE. If there is not enough information to find a given value or determine whether it exists, write NEI.
You do not need to show work. As a reminder, $g(x)$ is an odd function.

$g(g(3) - 1) =$ -1

$\lim_{x \to 3^+} g(x) =$ 1

$\lim_{x \to 4} g(x) =$ -1

$\lim_{x \to -3^+} g(x) =$ -3

$\lim_{x \to 2} g(x) =$ **DNE**

$\lim_{h \to 0} \dfrac{g(3.5 + h) - g(3.5)}{h} =$ -2

$\lim_{x \to 3^-} g(x) =$ 3

$\lim_{x \to \infty} g(x) =$ -2

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2. [8 points] Alana produces a range of kitchenware to honor her favorite comic book writers. Her new "Stan Lee" cups have been especially popular. Let $P(m)$ represent her profit, in thousands of dollars, if she produces m thousand cups, and let $F(m)$ represent the number of followers, in thousands, she will have on social media after she produces m thousand cups. Assume that both functions are invertible and differentiable.

a. [2 points] Write a complete sentence that gives a practical interpretation of the equation

$$F^{-1}(200) = 8.$$

Solution: Alana has 200,000 followers on social media after producing 8,000 cups.

b. [2 points] Write a mathematical equation using the functions P, F, and/or their inverses that represents the following statement.

If Alana makes a profit of 30 thousand dollars, she will have 250 thousand followers.

Solution: $F\big(P^{-1}(30)\big) = 250$.

c. [2 points] Complete the following sentence to give a practical interpretation of the equation

$$F'(10) = 32.$$

If Alana produces 12 thousand cups instead of producing 10 thousand cups, . . .

Solution: . . . she will have about 64,000 more followers.

d. [2 points] Circle the **one** statement below that is best supported by the equation

$$(P^{-1})'(16) = 4.$$

i. For every 4 thousand cups Alana produces, she makes an extra 16 thousand dollars in profit, roughly.

[✓ ii.] If Alana has made 15.5 thousand dollars in profit, and would like to make 500 more dollars in profit, she will need to produce about 2 thousand more cups.

iii. If Alana goes from producing 16 thousand cups to producing 17 thousand cups then her profit will increase by about 4 thousand dollars.

iv. If Alana produces an extra 500 cups after producing her first 4 thousand cups, then she will make about an extra 8 thousand dollars in profit.

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3. [5 points] Let

$$K(u) = \arctan(u^2 + 3u).$$

Use the limit definition of the derivative to write an explicit expression for $K'(2)$. *Your answer should not involve the letter K. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Solution:

$$K'(2) = \lim_{h\to 0} \frac{K(2+h) - K(2)}{h} = \lim_{h\to 0} \frac{\arctan\left((2+h)^2 + 3(2+h)\right) - \arctan(2^2 + 3\cdot 2)}{h}$$

$$= \lim_{h\to 0} \frac{\arctan\left(h^2 + 7h + 10\right) - \arctan(10)}{h}.$$

Answer: $K'(2) = \boxed{\lim_{h\to 0} \frac{\arctan\left((2+h)^2 + 3(2+h)\right) - \arctan(2^2 + 3\cdot 2)}{h}}$

4. [6 points] Suppose $b(x)$ is a differentiable function whose tangent line at the point $x = 4$ is given by the linear function $T(x)$. To the right is a table consisting of some values of $b(x)$ and $b'(x)$.

x	-3	-2	0	4
$b(x)$	5	1	-3	-6
$b'(x)$	-4	?	?	-1
$T(x)$	1	0	-2	-6

a. [2 points] Find the values of $T(x)$ at $x = -3, -2, 0$, and 4, and write them into the table.

Solution: We use the fact that $T(x) = b(4) + b'(4)(x-4) = -6 - (x-4) = -2 - x$.

b. [2 points] Use the table to estimate $b'(-1)$.

Solution: $b'(-1) \approx \frac{b(0) - b(-2)}{0 - (-2)} = \frac{-3 - 1}{2} = \frac{-4}{2} = -2.$

Answer: -2

c. [2 points] Find an equation for the line tangent to the graph of $y = b(x)$ at the point $(-3, 5)$.

Solution: This line has equation

$$y = b(-3) + b'(-3)(x - (-3)) = 5 - 4(x+3) = -4x - 7.$$

Answer: $y =$ $5 - 4(x+3)$

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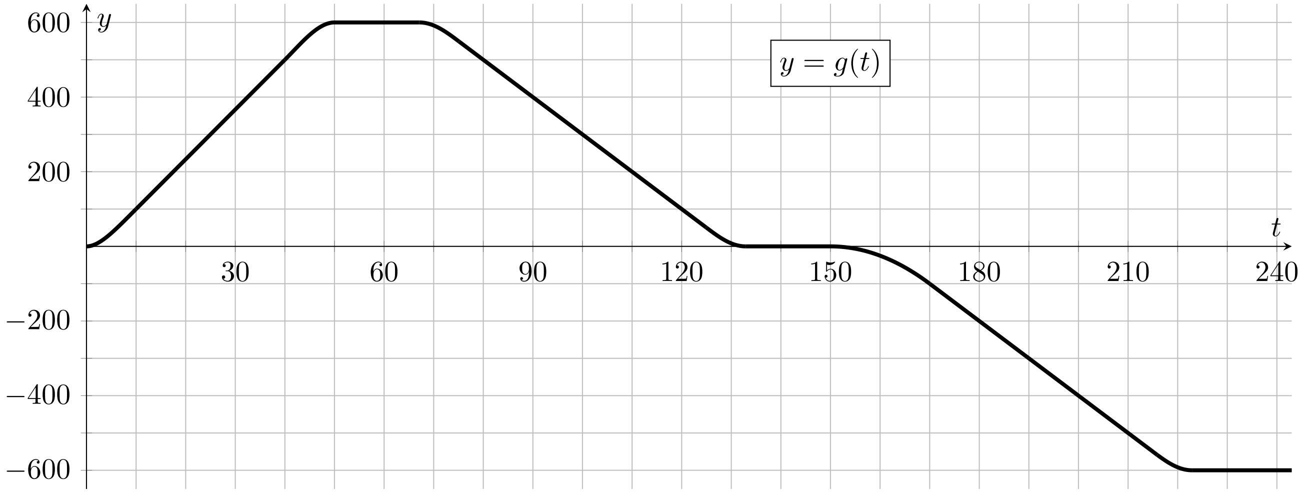
5. [9 points] The ExpressTram in the Detroit Metro Airport travels along a straight line through McNamara Terminal between the North and South Stations, with the Terminal Station in between. North and South Station are 600 meters north and 600 meters south, respectively, of the centrally located Terminal Station.

Suppose $g(t)$ is the position of the tram, in meters north of Terminal Station, t seconds after it leaves Terminal Station at 1pm on a certain day. A portion of the graph of $g(t)$ is given below.



a. [2 points] What is the total distance the tram traveled between 1:00pm and 1:02pm?

Solution: In the first 2 minutes (= 120 seconds) after 1pm, the tram travels 600 meters north, pauses for a bit, and then travels 500 meters south, for a total of 1100 meters.

Answer: **1100 meters** .

b. [2 points] It appears that the tram traveled at two different top speeds, depending on whether it was heading northward or southward. In which direction did it travel *faster*? Circle NORTH or SOUTH below. Then find the fastest speed at which the tram traveled, in meters per second.

Solution: The tram's speed going north is the slope of the linear segment from $t = 10$ to $t = 40$, which is $\frac{40}{3}$ m/s. The tram's speed going south is the absolute value of the slope of the linear segment from $t = 80$ to $t = 120$, which is 10 m/s.

[✓ NORTH] SOUTH **Answer:** 40/3 meters per second

c. [2 points] What was the average velocity of the tram over the time interval $10 \le t \le 210$? Remember that the velocity is positive when the tram is traveling north.

Solution: $\frac{g(210)-g(10)}{210-10} = \frac{-500-100}{200} = \frac{-600}{200} = -3$ meters per second.

Answer: -3 meters per second

d. [1 point] At which times in the interval $10 \le t \le 210$ was the tram's instantaneous velocity the *same* as its average velocity between $t = 10$ and $t = 210$? *Circle the one best answer.*

$t = 10$ $t = 49$ $t = 93$ [✓ $t = 156$] $t = 200$

e. [2 points] You are trying to get to Gate A64, which is 500 meters north of Terminal Station. Unfortunately, you just miss the ExpressTram at 1pm, so you start walking from Terminal Station towards your gate just as the tram departs. Assuming you walk at 3 meters per second, how far away from Gate A64 are you when the tram passes you again on its return trip?

Solution: The tram passes you again at the point where the graph of $g(t)$ meets the line of slope 3 through the origin, which is at $(100, 300)$. So at that moment you are 300 meters north of Terminal Station, and 200 meters south of Gate A64.

Answer: 200 meters away from Gate A64

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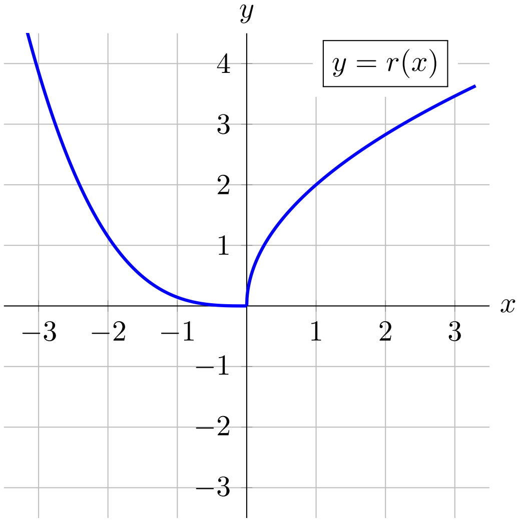
6. [9 points]

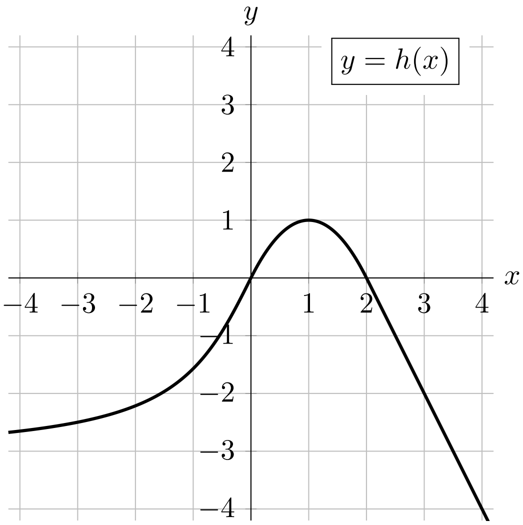
a. [5 points] Carefully draw the graph of a single function on the given axes that satisfies the given conditions.

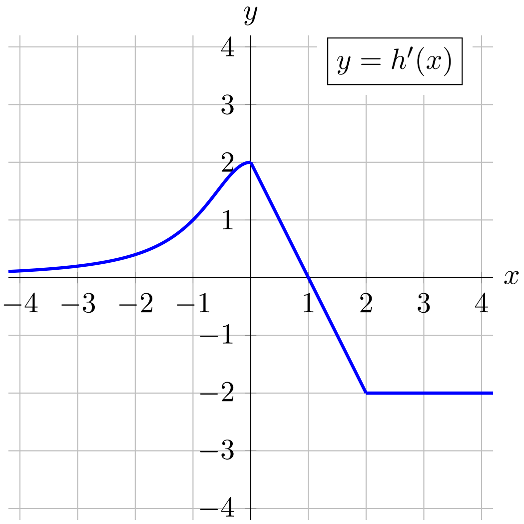
A continuous function $r(x)$ with domain containing $(-3,3)$ such that

- $r(x)$ decreasing and concave up on $(-3,0)$,
- $r(x)$ is increasing and concave down on $(0,3)$,
- $\displaystyle\lim_{h\to 0^-}\frac{r(h)-r(0)}{h}=0$,
- $\displaystyle\lim_{h\to 0^+}\frac{r(h)-r(0)}{h}=\infty$.



b. [4 points] A portion of the graph of the function $h(x)$ is shown below on the left. Note that $h(x)$ is linear for $x > 2$. Carefully sketch the graph of $h'(x)$ for $-4 < x < 4$ on the given axes on the right.





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7. [7 points] Consider the rational function $q(x) = \dfrac{5x(x+1)(x+3)^2}{(x-2)(x+1)^2(x+3)}$.

a. [2 points] Find all x-values at which the function $q(x)$ has a vertical asymptote.

Solution: After canceling common factors from the numerator and denominator to eliminate potential holes, we find that aside from a hole at $x = -3$, $q(x)$ has the same graph as

$$\frac{5x(x+3)}{(x-2)(x+1)},$$

which has vertical asymptotes at $x = 2$ and $x = -1$.

Answer: $q(x)$ has vertical asymptotes at $x =$ $2, -1$

b. [2 points] Find the following limits. If a limit diverges to ∞ or $-\infty$ or does not exist for any other reason, write DNE.

i. $\lim\limits_{x\to\infty} q(x)$

Solution: Since the numerator and denominator of $q(x)$ have the same degree, $\lim\limits_{x\to\infty} q(x)$ is the ratio of the leading coefficients of the numerator and denominator, which is $\frac{5}{1} = 5$.

Answer: 5

ii. $\lim\limits_{x\to -3} q(x)$

Solution: $\lim\limits_{x\to-3} q(x) = \lim\limits_{x\to-3} \dfrac{5x(x+3)}{(x-2)(x+1)} = 0.$

Answer: 0

Suppose the piecewise function $g(x)$ is defined as follows, where $q(x)$ is as above, and k is a constant.

$$g(x) = \begin{cases} e - e^{kx^3} & x \le 1 \\ q(x) & x > 1 \end{cases}$$

c. [3 points] Find an *exact* value of k for which the function $g(x)$ is continuous at $x = 1$. Show your work.

Solution: The function $g(x)$ is continuous at $x = 1$ if $\lim\limits_{x\to1} g(x) = g(1)$. Since

$$g(1) = e - e^k = \lim_{x\to1^-}\left(e - e^{kx^3}\right) = \lim_{x\to1^-} g(x),$$

we just need to find k such that $e - e^k = \lim\limits_{x\to1^+} g(x) = \lim\limits_{x\to1^+} q(x) = q(1) = \dfrac{5\cdot2\cdot16}{(-1)\cdot4\cdot4} = -10.$

Now we solve:

$$\begin{aligned} e - e^k &= -10 \\ e + 10 &= e^k \\ \ln(e+10) &= k. \end{aligned}$$

Answer: $k =$ $\ln(e+10)$

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8. [7 points] Gwen and Miles are building sand castles on a beach where the water level varies sinusoidally with the tides over time.

At 9am, when the water level is decreasing, they mark a spot on the beach that is right at the water level at that time. Over the next few hours they build a giant castle on that spot, wondering when the water will return to the base of the castle and start to destroy it. As it turns out, the water reaches the castle again for the first time three hours later, at 12pm.

a. [3 points] Based on the information above, determine which of the following sinusoidal functions could possibly model the number of meters that the water level is above the base of the sand castle t hours after 9am. *Circle all correct answers.*

i. $\sin\left(\frac{\pi}{3}t\right)$

ii. [✓ $-\sin\left(\frac{\pi}{3}t\right)$]

iii. $\cos\left(\frac{\pi}{3}t\right)$

iv. $-\cos\left(\frac{\pi}{3}t\right)$

v. [✓ $-\cos\left(\frac{2\pi}{12.5}(t-1.5)\right)+\cos\left(\frac{2\pi}{12.5}(-1.5)\right)$]

vi. $-\sin\left(\frac{2\pi}{12.5}(t-1.5)\right)+\sin\left(\frac{2\pi}{12.5}(-1.5)\right)$

vii. NONE

b. [2 points] Assuming $f(t)$ is the function which gives the height, in meters, of the water level above the base of the sandcastle t hours after 9am, which function below gives the height in meters of the water above the base of the sand castle m minutes after 10am? *Circle the one correct answer.*

i. [✓ $f\left(\frac{m+60}{60}\right)$]

ii. $f(60(m-1))$

iii. $60f(m+1)$

iv. $\frac{f(m-1)}{60}$

v. $f\left(\frac{m+1}{60}\right)$

vi. NONE

c. [2 points] The next day, which is a Tuesday, Gwen and Miles return to the spot where they had built their sand castle the day before, and they observe that today the tide reaches that spot at 1pm, while the water level is increasing. Circle all times below when the water level is exactly at the point where they built their sand castle. *Circle all correct answers.*

i. 9am Tue

ii. [✓ 10am Tue]

iii. 12pm Tue

iv. 4pm Tue

v. [✓ 11am Wed]

vi. 1pm Wed

vii. NONE

Solution: Since there was a three hour gap on Monday between the receding water passing the sand castle and the rising water returning to the same point, this must be true on Tuesday as well, so the water level must have been at the level of the sand castle three hours prior to 1pm, at 10am Tuesday. So the receding water passed the point with the sand castle at 9am Monday and 10am Tuesday, which means the same must happen again 25 hours later at 11am on Wednesday. There are no other times listed when we can know for certain the the water will be at the level of the sand castle.

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