Math 115 — Second Midterm — April 2, 2024

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 10 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

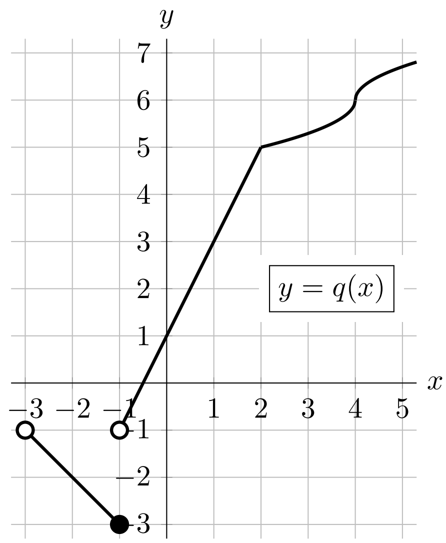
Problem	Points	Score
1	9	
2	10	
3	9	
4	8	
5	4	

Problem	Points	Score
6	4	
7	12	
8	8	
9	7	
10	9	
Total	80	

1. [9 points]

A portion of the graph of the invertible function $q(x)$ is shown to the right. Note that:

- $q(x)$ is linear on $(-3,-1]$ and on $(-1,2]$.
- There is a corner at $x=2$.
- The tangent line to $q(x)$ at $x=4$ is vertical.



For parts **a.–c.**, find the **exact** values, or write NEI if there is not enough information to do so, or write DNE if the value does not exist. Your answers should not include the letter q, but you do not need to simplify. *Show work.*

a. [2 points] Let $A(x) = q^{-1}(x)$. Find $A'(4)$.

Solution: $A'(4) = \left(q^{-1}\right)'(4) = \dfrac{1}{q'\left(q^{-1}(4)\right)} = \dfrac{1}{q'(1.5)} = \dfrac{1}{2}.$

Answer: $A'(4) =$ $1/2$

b. [2 points] Let $B(x) = \dfrac{x}{q(x)}$. Find $B'(-2)$.

Solution: By the Quotient Rule, $B'(x) = \dfrac{q(x) - xq'(x)}{q(x)^2}$, so

$$B'(-2) = \frac{q(-2) + 2q'(-2)}{q(-2)^2} = \frac{-2 + 2(-1)}{(-2)^2} = \frac{-4}{4} = -1.$$

Answer: $B'(-2) =$ -1

c. [3 points] Let $C(x) = \cos\left(\dfrac{\pi}{2}xq(x)\right)$. Find $C'(1)$.

Solution: Using the Chain Rule and Product Rule, we have

$$C'(x) = -\sin\left(\frac{\pi}{2}xq(x)\right) \cdot \left(\frac{\pi}{2}q(x) + \frac{\pi}{2}xq'(x)\right).$$

Since $q(1) = 3$ and $q'(1) = 2$, this means

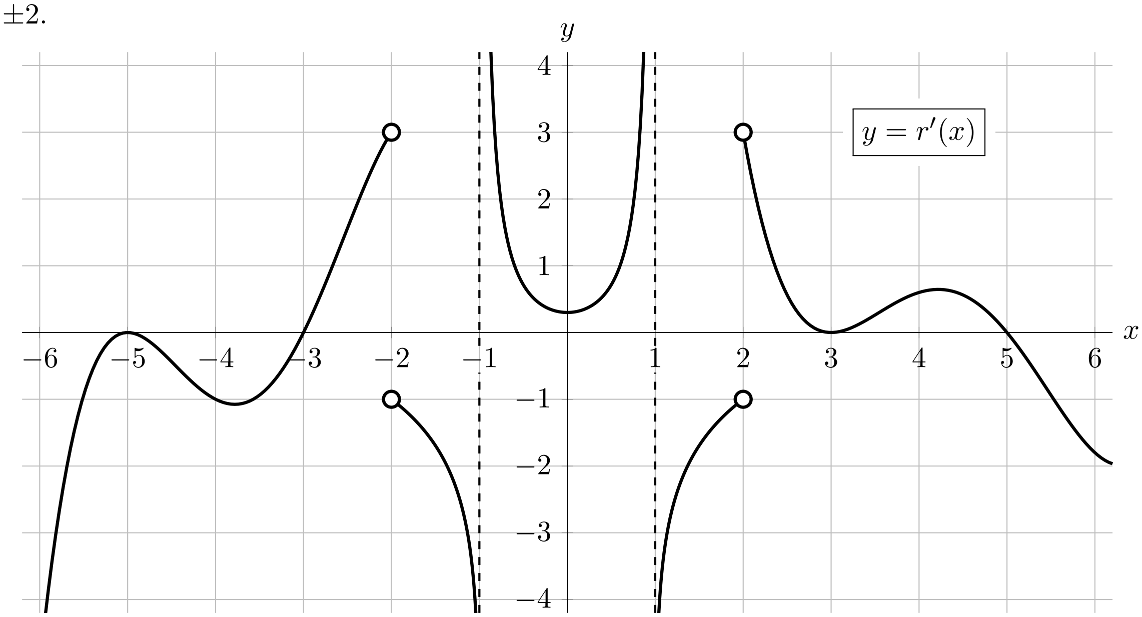
$$C'(1) = -\sin\left(\frac{3\pi}{2}\right) \cdot \left(\frac{3\pi}{2} + \pi\right) = -\frac{5\pi}{2}\sin\left(\frac{3\pi}{2}\right) = \frac{5\pi}{2}.$$

Answer: $C'(1) =$ $\dfrac{5\pi}{2}$

d. [2 points] On which of the following intervals does $q(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers. You do not need to show work for this part.

$[-1,1]$ $\quad$ $\boxed{[2,3.5]}$ $\quad$ $[3,5]$ $\quad$ NONE OF THESE

2. [10 points] Suppose $r(x)$ is a continuous function, defined for all real numbers. A portion of the graph of $r'(x)$, the **derivative** of $r(x)$, is given below. Assume $r'(x)$ is differentiable everywhere $r'(x)$ is defined. Note that $r'(x)$ has vertical asymptotes at $x = \pm 1$, and is undefined at $x = \pm 1$ and $x = \pm 2$.



a. [2 points] Circle all points below that are critical points of $r(x)$.

$x = 0$ $\quad$ $\boxed{x = 1}$ $\quad$ $\boxed{x = 2}$ $\quad$ $\boxed{x = 3}$ $\quad$ $\boxed{x = 5}$ $\quad$ NONE OF THESE

b. [2 points] Circle all intervals below on which $r(x)$ is increasing on the entire interval.

$\boxed{(-3, -2)}$ $\quad$ $(-2, -1)$ $\quad$ $\boxed{(-1, 1)}$ $\quad$ $(1, 2)$ $\quad$ $\boxed{(2, 3)}$ $\quad$ NONE OF THESE

c. [2 points] Circle all intervals below on which $r(x)$ is concave up on the entire interval.

$\boxed{(-3, -2)}$ $\quad$ $(-2, -1)$ $\quad$ $(-1, 1)$ $\quad$ $\boxed{(1, 2)}$ $\quad$ $(2, 3)$ $\quad$ NONE OF THESE

d. [2 points] Circle all intervals below in which $r(x)$ has a local minimum.

$\boxed{(-4, -2)}$ $\quad$ $(0, 2)$ $\quad$ $\boxed{(1, 3)}$ $\quad$ $(2, 4)$ $\quad$ $(4, 6)$ $\quad$ NONE OF THESE

e. [2 points] Circle all intervals below in which $r(x)$ has a local maximum.

$(-4, -2)$ $\quad$ $\boxed{(0, 2)}$ $\quad$ $(1, 3)$ $\quad$ $(2, 4)$ $\quad$ $\boxed{(4, 6)}$ $\quad$ NONE OF THESE

3. [9 points] Suppose

$$g(x) = \sqrt{x^2+1} \qquad \text{and} \qquad h(x) = ke^{2x}\ln x,$$

where k is a real number constant. Note that

$$g'(x) = \frac{x}{\sqrt{x^2+1}} \qquad \text{and} \qquad g''(x) = (x^2+1)^{-3/2}.$$

a. [2 points] Find a formula for the linear approximation $L(x)$ of the function $g(x)$ at the point $x = -1$. Your answer should not include the letter g, but you do not need to simplify.

Solution:

$$L(x) \;=\; g(-1) + g'(-1)(x+1) \;=\; \sqrt{2} - \frac{1}{\sqrt{2}}(x+1).$$

Answer: $L(x) =$ $\sqrt{2} - \frac{1}{\sqrt{2}}(x+1)$

b. [1 point] Does $L(x)$ give an overestimate or underestimate for $g(x)$ near $x = -1$? Circle your answer below. *No justification needed.*

[UNDERESTIMATE] OVERESTIMATE

c. [3 points] Find a formula for $h'(x)$. Your answer may include the constant k.

Solution: Using the Product Rule and the Chain Rule, we get:

Answer: $h'(x) =$ $2ke^{2x}\ln x + \dfrac{ke^{2x}}{x}$

d. [3 points] Find a value of k for which the function

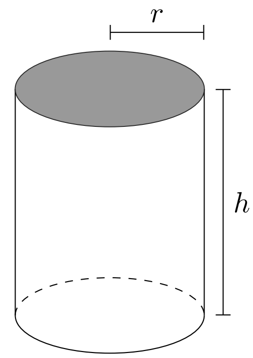
$$f(x) = \begin{cases} g(x) & x \le 1 \\ h(x) & x > 1 \end{cases}$$

is differentiable, if this is possible. If no such value of k exists, write DNE on the answer line and briefly justify your answer. *Show your work.* [Note: Recall that $\ln(1) = 0$.]

Solution: Since $g(1) = \sqrt{2}$ and $h(1) = ke^2 \ln 1 = 0$, we have $g(1) \neq h(1)$ no matter what k is, so there is no value of k that makes $f(x)$ continuous at $x = 1$, and therefore no value of k that makes $f(x)$ differentiable at $x = 1$.

Answer: $k =$ **DNE**

4. [8 points]



Alana is designing a new prototype for her Stan Lee cups. The new cups are cylindrical in shape, with metal sides and base, and a circular lid made from silicone. If the cylinder has height h centimeters, and radius r centimeters, then the surface area of the metal part is $2\pi rh + \pi r^2$ square centimeters, and the surface area of the silicone part is πr^2. The metal costs 2 cents per square centimeter, and the silicone costs 3 cents per square centimeter. Alana spends a total of 300 cents on materials for each cup.

a. [3 points] Find a formula for h in terms of r.

Solution: The total cost in cents of materials for each cup is

$$300 \;=\; 2\big(2\pi rh + \pi r^2\big) + 3\pi r^2 \;=\; 4\pi rh + 5\pi r^2.$$

Therefore, $4\pi rh = 300 - 5\pi r^2$, so $h = \dfrac{300 - 5\pi r^2}{4\pi r} = \dfrac{75}{\pi r} - \dfrac{5r}{4}$.

Answer: $h = \dfrac{300 - 5\pi r^2}{4\pi r}$

b. [1 point] Recall that the volume of a cylinder of radius r and height h is $V = \pi r^2 h$. Write a formula for $V(r)$, the volume of one of the cups in cubic centimeters, as a function of r only. *Your formula should not include the letter h.*

Answer: $V(r) = \pi r^2 \cdot \dfrac{300 - 5\pi r^2}{4\pi r} = r \cdot \dfrac{300 - 5\pi r^2}{4}$

c. [4 points] Alana wants to ensure that the height of a cup is at most 2 and a half times its radius, that is, she wants $h \le 2.5r$. Given this constraint, find the domain of $V(r)$ in the context of this problem.

Solution: In the context of this problem, the height should be positive, so we have the constraints $0 < h \le 2.5r$. Substituting in our expression for h from part **a.**, we get:

$$\begin{aligned} 0 \;&<\; \frac{300 - 5\pi r^2}{4\pi r} \;\le\; 2.5r \\ 0 \;&<\; 300 - 5\pi r^2 \;\le\; 10\pi r^2 \\ 5\pi r^2 \;&<\; 300 \;\le\; 15\pi r^2. \end{aligned}$$

Therefore, we must have

$$\frac{300}{15\pi} \;\le\; r^2 \;<\; \frac{300}{5\pi}, \qquad \text{which means} \qquad \sqrt{\frac{20}{\pi}} \;\le\; r \;<\; \sqrt{\frac{60}{\pi}}.$$

Answer: $\left[\sqrt{\dfrac{20}{\pi}}, \sqrt{\dfrac{60}{\pi}}\right)$

5. [4 points] One of Alana's interns suggests that instead of trying to fix the cost of their Stan Lee cups and maximize the volume, they should instead fix the volume and try to minimize the cost. Assuming they try to match their competitor's standard cup size of 500ml, their cost of producing each cup is

$$C(r) = \frac{2000}{r} + 5\pi r^2 \quad \text{dollars},$$

where r is the radius of the cup. Assuming the only constraint on r is that $r > 0$, what is the cup radius that minimizes their cost? *Show all your work.*

Solution: We need to minimize $C(r)$ over the domain $(0, \infty)$. Taking a derivative, we find

$$C'(r) = -2000r^{-2} + 10\pi r,$$

so $C(r)$ is differentiable everywhere on $(0, \infty)$. Setting $C'(r) = 0$ and solving we get:

$$\frac{2000}{r^2} = 10\pi r, \quad 2000 = 10\pi r^3, \quad \frac{200}{\pi} = r^3,$$

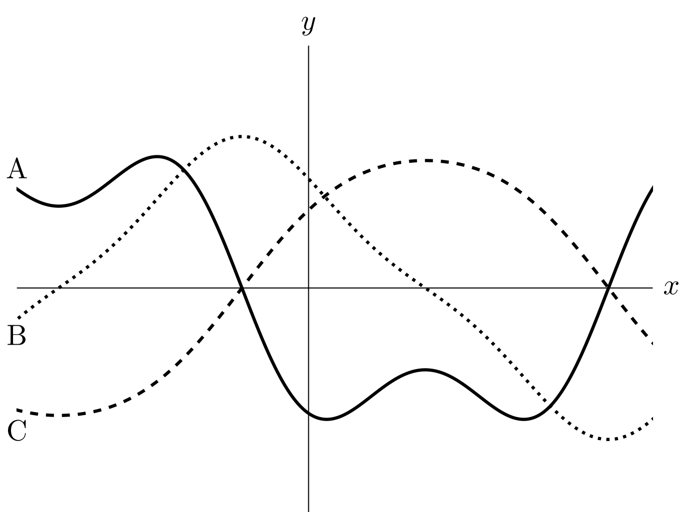
so the only critical point of $C(r)$ on $(0, \infty)$ is $r = \sqrt[3]{\frac{200}{\pi}} = \left(\frac{200}{\pi}\right)^{1/3}$. Since

$$\lim_{x\to 0^+} C(r) = \infty = \lim_{x\to\infty} C(r),$$

this critical point must indeed be the global minimum of $C(r)$ on $(0, \infty)$.

Answer: $r =$ $\left(\frac{200}{\pi}\right)^{1/3}$ centimeters

6. [4 points] Shown below are portions of the graphs of the functions $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.



Answer: $f(x)$: C

$f'(x)$: B

$f''(x)$: A

7. [12 points] Let $f(x) = x^4e^x + 4e$ and $g(x) = -x^2 + (2+5e)x - 1$, and let $h(x)$ be the piecewise function

$$h(x) = \begin{cases} f(x) & x \leq 1 \\ g(x) & x > 1. \end{cases}$$

Note that $f(1) = 5e = g(1)$ and $f'(1) = 5e = g'(1)$, so $h(x)$ is continuous and differentiable at $x = 1$. To answer the questions below, you may use the following:

$$f'(x) = x^3e^x(x+4) \qquad \text{and} \qquad f''(x) = x^2e^x(x+2)(x+6).$$

a. [3 points] Find all critical points of $h(x)$. *No justification necessary.*

Solution: We are given that $h'(1)$ exists and is nonzero, so $x = 1$ is not a critical point of $h(x)$. This means the critical points of $h(x)$ are the critical points of $f(x)$ in $(-\infty, 1)$ together with the critical points of $g(x)$ in $(1, \infty)$. This includes $x = -4$, $x = 0$, and also $x = 1 + \frac{5e}{2}$ since $1 + \frac{5e}{2} > 1$.

Answer: $h(x)$ has critical points at $x =$ $-4, \quad 0, \quad 1 + \dfrac{5e}{2}$

b. [3 points] Find all critical points of $h'(x)$. *No justification necessary.*

Solution: Since $g'(x)$ has no critical points on $(1, \infty)$, the critical points of $h'(x)$ will be the critical points of $f'(x)$ on $(-\infty, 1)$, together with $x = 1$ if $h''(1)$ is zero or undefined. Since $f''(1) = 21e \neq -2 = g''(1)$, we see that $h''(1)$ does not exist, so $x = 1$ is a critical point of $h'(x)$. The other critical points of $h'(x)$ are the critical points of $f'(x)$ on $(-\infty, 1)$, which are $x = -6$, $x = -2$, and $x = 0$.

Answer: $h'(x)$ has critical points at $x =$ $-6, \quad -2, \quad 0, \quad 1$

c. [6 points] Find all inflection points of $h(x)$. *Show all your work. Be sure you* <u>*show enough evidence*</u> *to justify your conclusions.*

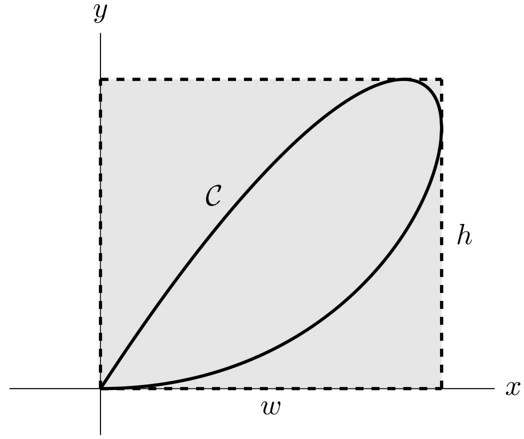
Solution: The inflection points of $h(x)$ occur at x-values where the graph of $h(x)$ changes concavity, that is, where $h''(x)$ changes sign from positive to negative or vice versa. So we need to determine the sign of $h''(x)$ on the five intervals determined by the four critical points of $h'(x)$ that we found in part **b**.

For $x < 1$, we have $h''(x) = f''(x)$, where $f''(x)$ is a product of the four terms x^2, e^x, $(x+2)$, and $(x+6)$, the first two of which are positive in each interval. For $x > 1$, we have $h''(x) = g''(x) = -2 < 0$. This produces the following sign chart for $h''(x)$:

$h''(x)$:	$x<-6$	-6	$-6<x<-2$	-2	$-2<x<0$	0	$0<x<1$	1	$x>1$
sign	$++--=+$		$++-+=-$		$++++=+$		$++++=+$		$-$
concavity	∪		∩		∪		∪		∩

The graph of $h(x)$ is concave up on intervals where $h''(x) > 0$, and concave down on intervals where $h''(x) < 0$, so the inflection points of $h(x)$ are at $x = -6$, $x = -2$, and $x = 1$, where $h''(x)$ changes sign. The critical point $x = 0$ of $h'(x)$ is not an inflection point of $h(x)$, since $h(x)$ is concave up on both $(-2, 0)$ and $(0, 1)$.

Answer: $h(x)$ has inflection points at $x =$ $-6, \quad -2, \quad 1$



8. [8 points]

Let $\mathcal{C}$ be the curve implicitly defined by the equation

$$x^3 - 3xy + y^2 = 0.$$

The portion of the curve $\mathcal{C}$ that lies in the first quadrant is pictured to the right, not necessarily to scale, along with the smallest possible rectangle that contains it and has sides on the coordinate axes. This rectangle is shaded, and has side lengths w and h.

a. [4 points] Use implicit differentiation to find $\frac{dy}{dx}$.

Solution: Implicitly differentiating the given equation with respect to x, we get:

$$\begin{aligned} 3x^2 - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0, \\ \frac{dy}{dx}(2y - 3x) &= 3y - 3x^2, \\ \frac{dy}{dx} &= \frac{3(y - x^2)}{2y - 3x}. \end{aligned}$$

Answer: $\frac{dy}{dx} =$ $\frac{3(y-x^2)}{2y-3x}$

b. [4 points] Find w and h, the width and height of the shaded rectangle containing $\mathcal{C}$. *Show all your work.*

Solution: The width w is the x-coordinate of the point on $\mathcal{C}$ with a vertical tangent line, and the height h is the y-coordinate of one of the points on $\mathcal{C}$ with a horizontal tangent line. Since $\frac{dy}{dx}$ is undefined where $\mathcal{C}$ has a vertical tangent line, and zero where $\mathcal{C}$ has a horizontal tangent line, we can find w and h, respectively, by setting the numerator and denominator of $\frac{dy}{dx}$ equal to zero and solving.

$$\frac{dy}{dx} = 0 \implies y = x^2 \implies x^3 - 3x(x^2) + (x^2)^2 = 0$$
$$\implies x^4 - 2x^3 = x^3(x-2) = 0.$$

This means $\mathcal{C}$ has a horizontal tangent line at $x = 2$ and $x = 0$, and from the picture we see that h is the y-coordinate of the point on $\mathcal{C}$ with x-coordinate 2. Plugging $x = 2$ into $y = x^2$, we get $h = y = 4$.

To find w, we find points on $\mathcal{C}$ where $\frac{dx}{dy} = 0$, or equivalently where $\frac{dy}{dx}$ is undefined. This happens when $y = \frac{3}{2}x$. Substituting $y = \frac{3}{2}x$ into the equation $x^3 - 3xy + y^2 = 0$, we get

$$0 = x^3 - 3x\left(\frac{3}{2}x\right) + \left(\frac{3}{2}x\right)^2 = x^3 - \frac{9}{2}x^2 + \frac{9}{4}x^2 = x^2\left(x - \frac{9}{4}\right).$$

Thus $\mathcal{C}$ has a vertical tangent line at $\left(\frac{9}{4}, \frac{27}{8}\right)$, so we get $w = \frac{9}{4}$ as the width of the rectangle.

Answer: $w =$ 9/4 and $h =$ 4

9. [7 points] Alana heats water using her "Simone Steamer," a peculiar kettle with the face of comic book writer Gail Simone on it, and tracks the depth of water as she fills it up.

a. [2 points] The depth of the water H, in cm, t seconds after Alana starts filling the kettle is given by $H = f(t)$. Circle the **one** statement below that is best supported by the equation

$$(f^{-1})'(5) = 2.$$

i. After Alana has been filling the kettle for 5 seconds, the depth of water will increase by about 1 cm in the next half-second.

ii. It takes approximately one second for the depth of water to increase from 4.5 to 5cm.

iii. Every two seconds, the depth of water increases by about 5cm.

iv. After Alana has been filling the kettle for 2 seconds, the depth of water is roughly 5cm.

b. [2 points] The volume V, in cm^3, of water in the kettle is related to the depth H by the formula

$$V = \frac{1}{5}\left(H^3 + 4H^2 + 10H\right).$$

Find an expression for $\frac{dV}{dt}$ in terms of H and $\frac{dH}{dt}$.

Solution: Implicitly differentiating each side with respect to t, we get

$$\frac{dV}{dt} = \frac{1}{5}\left(3H^2\frac{dH}{dt} + 8H\frac{dH}{dt} + 10\frac{dH}{dt}\right) = \left(\frac{3}{5}H^2 + \frac{8}{5}H + 2\right)\frac{dH}{dt}.$$

Answer: $\frac{dV}{dt} = \left(\frac{3}{5}H^2 + \frac{8}{5}H + 2\right)\frac{dH}{dt}$

c. [3 points] When Alana has been filling the kettle for 3 seconds, the depth of the water is 5cm. Use this and the fact (from part **a.**) that $(f^{-1})'(5) = 2$ to determine the rate at which the **volume** of water in the kettle is increasing at $t = 3$. *Show all your work.*

Solution: We must find $\frac{dV}{dt}$ when $t = 3$. We can find this using our answer to part **b.**, provided we can find H and $\frac{dH}{dt}$ when $t = 3$. We are given that the depth of the water when $t = 3$ is $H = f(3) = 5$ cm. And when $t = 3$, the rate $\frac{dH}{dt}$ at which the depth of the water is changing is

$$f'(3) = \frac{1}{(f^{-1})'(f(3))} = \frac{1}{(f^{-1})'(5)} = \frac{1}{2}.$$

So, when $t = 3$, we have

$$\frac{dV}{dt} = \left(\frac{3}{5}(5)^2 + \frac{8}{5}(5) + 2\right)\cdot\frac{1}{2} = \frac{25}{2}.$$

Answer: The rate is: 25/2 cm^3 per second.

10. [9 points] A table of some values of the function $f(x)$ and its first and second derivatives is given below. The functions $f(x)$, $f'(x)$, and $f''(x)$ are continuous everywhere.

x	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	-0.6	0	-0.3	-2	-3	-1	0	3	88	204
$f'(x)$	3	0	-1	-2	0	2	0	9	80	0
$f''(x)$	-8	0	-2	0	4	0	0	22	0	-74

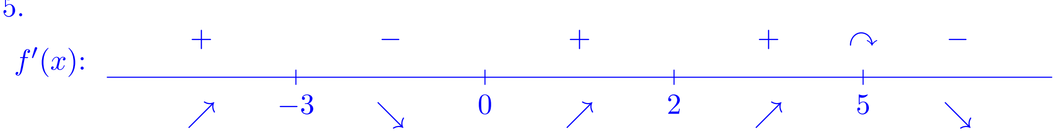
Assume that the critical points of $f(x)$ and $f'(x)$ are as follows, with no additional critical points besides those listed:

critical points of $f(x)$: $-3,\ 0,\ 2,\ 5$

critical points of $f'(x)$: $-3,\ -1,\ 1,\ 2,\ 4$

a. [4 points] Find all local extrema of $f(x)$, and classify each as a max or a min. If there are none of a particular type, write NONE. *No justification is necessary, although limited partial credit may be awarded for work shown.*

Solution: Using the table, we can produce a sign chart for $f'(x)$ as follows. Note that in order to determine the sign of $f'(x)$ on $(5, \infty)$, we can apply the Second Derivative Test at $x = 5$ to conclude that the graph of $f(x)$ is concave down near $x = 5$, so $f(x)$ has a local max at $x = 5$, so $f'(x) < 0$ for $x > 5$.



From this sign chart for $f'(x)$, the First Derivative Test tells us that $f(x)$ has a local min at $x = 0$, and local maxes as $x = -3$ and $x = 5$.

Answer: Local min(s) at $x =$ 0

Answer: Local max(es) at $x =$ -3, 5

b. [3 points] Find all global extrema of $f(x)$ on the interval $[-4, 3]$, and classify each as a max or a min. If there are none of a particular type, write NONE. *No justification is necessary, although limited partial credit may be awarded for work shown.*

Solution: We just need to check the values of $f(x)$ at the two endpoints and at all the critical points of f in the interval $(-4, 3)$. The largest of these values will be the global max of $f(x)$ on $[-4, 3]$, and the least will be the global min. We have:

$$f(-4) = -0.6, \quad f(-3) = 0, \quad f(0) = -3, \quad f(2) = 0, \quad f(3) = 3.$$

So $f(x)$ has a global min at $x = 0$ and a global max at $x = 3$ on the interval $[-4, 3]$.

Answer: Global min(s) at $x =$ 0

Answer: Global max(es) at $x =$ 3

c. [2 points] Circle all intervals below on which $f(x)$ must be <u>concave down</u> on the entire interval.

[$(-4, -2)$] [$(-3, -1)$] $(-1, 0)$ [$(4, 5)$] [$(5, \infty)$] NONE OF THESE