Math 115 — Final Exam — April 25, 2024

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 11 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	10	
3	10	
4	11	
5	11	

Problem	Points	Score
6	6	
7	6	
8	8	
9	11	
10	14	
Total	100	

1. [13 points] Given below is a table of values for a function $f(x)$ and its derivative $f'(x)$. The functions $f(x)$, $f'(x)$, and $f''(x)$ are all defined and continuous on $(-\infty, \infty)$.

x	0	2	4	6	8	10	12
$f(x)$	15	12	11	7	-2	3	5
$f'(x)$	-3	0	-2	-4	0	2	6

Assume that between consecutive values of x given in the table above, $f(x)$ is either **always increasing** or **always decreasing**.

In **a.–d.**, find the numerical value **exactly**, or write NEI if there is not enough information provided to do so. You do not need to simplify your numerical answers. *You do not need to show work on this page, but limited partial credit may be awarded for work shown.*

a. [2 points] Find $\lim_{s \to 0} \dfrac{f(4+s) - f(4)}{s}$.

Solution: By the definition of derivative, $\lim_{s \to 0} \dfrac{f(4+s) - f(4)}{s} = f'(4) = -2$.

Answer: -2

b. [2 points] If $B(x) = x^3 f(x)$, find $B'(10)$.

Solution: By the Product Rule, $B'(x) = 3x^2 f(x) + x^3 f'(x)$, so

$$B'(10) = 3 \cdot 10^2 \cdot f(10) + 10^3 \cdot f'(10) \;=\; 300 \cdot 3 + 1000 \cdot 2 \;=\; 2900.$$

Answer: 2900

c. [2 points] Find $\displaystyle\int_4^8 f'(x)\, dx$.

Solution: By the Fundamental Theorem of Calculus,

$$\int_4^8 f'(x)\, dx \;=\; f(8) - f(4) \;=\; -2 - 11 \;=\; -13.$$

Answer: -13

d. [2 points] Find $\displaystyle\int_0^{10} \left(5f''(x) - 3x^2\right)\, dx$.

Solution: By properties of integrals and the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_0^{10} \left(5f''(x) - 3x^2\right)\, dx &= 5\int_0^{10} f''(x)\, dx - 3\int_0^{10} x^2\, dx \\ &= 5\big(f'(10) - f'(0)\big) - 3\left(\tfrac{1}{3}x^3\Big]_0^{10}\right) \\ &= 5\big(2 - (-3)\big) - 3\big(1000/3\big) \;=\; 25 - 1000 \;=\; -975.\end{aligned}$$

Answer: -975

This problem continues on the next page.

This problem continues from the previous page. The problem statement is repeated for convenience.

Given below is a table of values for a function $f(x)$ and its derivative $f'(x)$. The functions $f(x)$, $f'(x)$, and $f''(x)$ are all defined and continuous on $(-\infty, \infty)$.

x	0	2	4	6	8	10	12
$f(x)$	15	12	11	7	-2	3	5
$f'(x)$	-3	0	-2	-4	0	2	6

Assume that between consecutive values of x given in the table above, $f(x)$ is either **always increasing** or **always decreasing**.

e. [2 points] Use a right Riemann sum with four equal subdivisions to estimate $\int_0^8 f(x)\,dx$. Write out all the terms in your sum. Your answer should not include the letter f, but you do not need to simplify.

Solution:

$$2\Big(f(2)+f(4)+f(6)+f(8)\Big) = 2(12+11+7-2) = 56.$$

f. [1 point] Does the answer to part **e.** overestimate, underestimate, or equal the value of $\int_0^8 f(x)\,dx$? Circle your answer. If there is not enough information to decide, circle NEI.

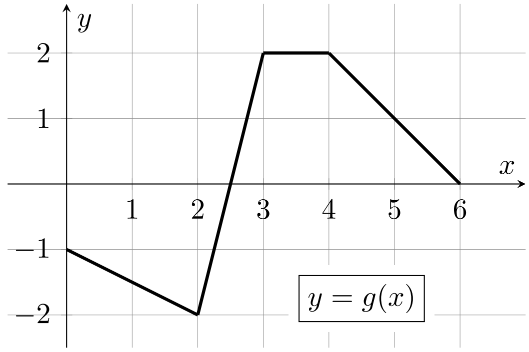
Answer: OVERESTIMATE UNDERESTIMATE EQUAL NEI

g. [2 points] Use a left Riemann sum with two equal subdivisions to estimate $\int_2^8 xf(2x)\,dx$. Write out all the terms in your sum. Your answer should not include the letter f, but you do not need to simplify.

Solution:

$$3\Big(2\cdot f(4)+5\cdot f(10)\Big) = 3\cdot(2\cdot 11+5\cdot 3) = 3\cdot 37 = 111.$$

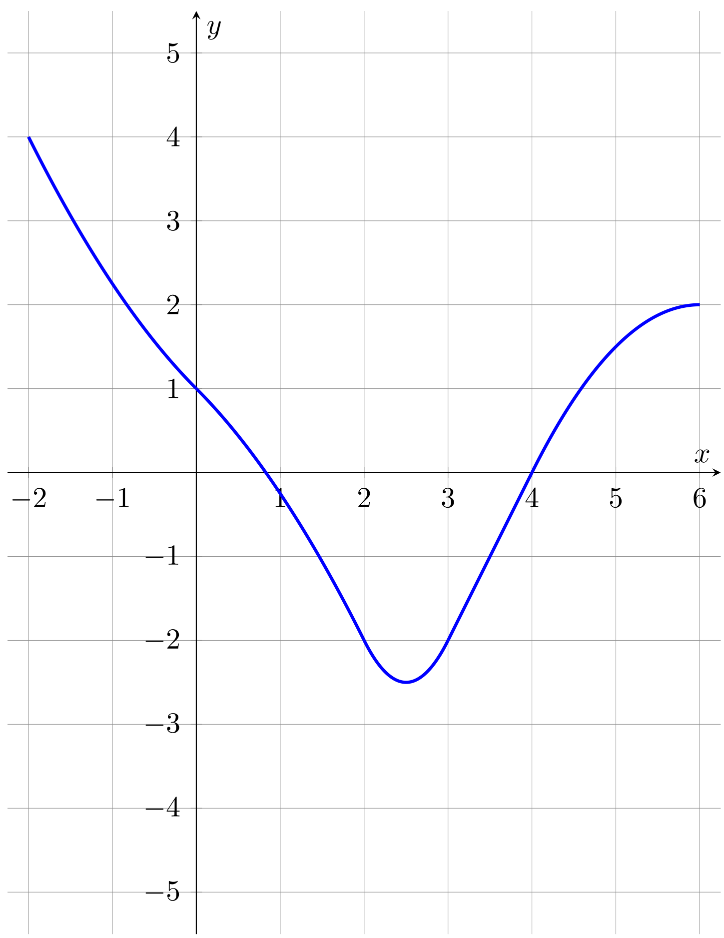
2. [10 points] An **even** function $g(x)$, which is defined for all real numbers, is graphed on the interval $[0, 6]$ below. Note that $g(x)$ is piecewise linear.



a. [4 points] The function $g(x)$ has a continuous antiderivative, $G(x)$, which satisfies $G(2) = -2$. Complete the following table of values for $G(x)$.

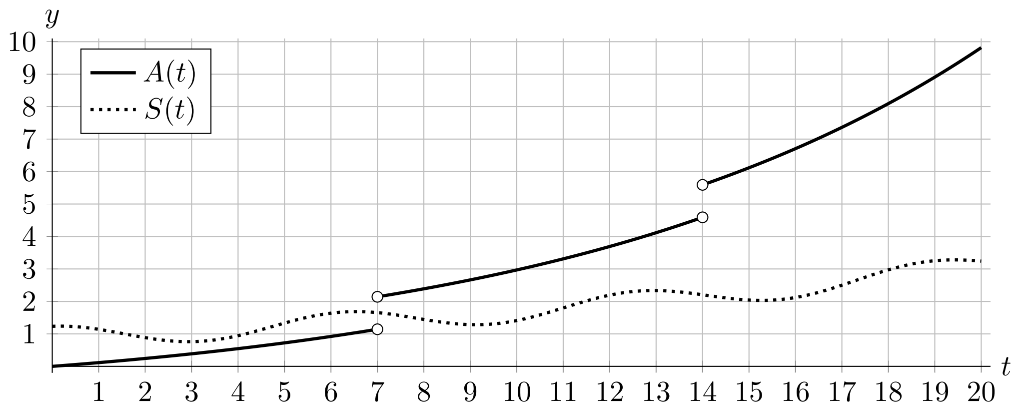
x	-2	0	2	4	6
$G(x)$	4	1	−2	0	2

b. [6 points] Sketch a graph of $G(x)$ on the interval $[-2, 6]$ using the axes provided. Make sure to clearly label the values at the points in your table above and also make it clear where $G(x)$ is increasing or decreasing, and where $G(x)$ is concave up, concave down, or linear.



3. [10 points] The San-Ti, inhabitants of a nearby star system, are terraforming an uninhabited planet which they are hoping to relocate to in the future. The planet is venting oxygen and does not yet have enough of it for them to survive, so they have placed large oxygen generators along with some vegetation on the planet's surface to increase the amount of oxygen in the atmosphere.

Let $A(t)$ be the rate at which oxygen is being added to the planet's atmosphere, and $S(t)$ the rate at which oxygen from the atmosphere is leaking into space, both measured in petagrams (Pg) per decade, t decades after the terraforming began. Graphs of $A(t)$ and $S(t)$ are shown below.



a. [2 points] Estimate the rate, in petagrams per decade, at which the amount of oxygen in the planet's atmosphere was changing 15 decades after the terraforming began.

Answer: 4 petagrams per decade.

b. [2 points] Write an expression for the total amount of oxygen, in petagrams, that leaked into space from the planet's atmosphere over the first 10 decades since the terraforming began. Your expression may involve one or more integrals.

Answer: $\int_0^{10} S(t)\,dt$ petagrams.

c. [2 points] Estimate the number of decades after terraforming began when there was the *least* amount of oxygen present in the planet's atmosphere.

Answer: 7 decades.

d. [2 points] Write an expression for the average rate of change in the amount of oxygen in the planet's atmosphere over the first 20 decades of the terraforming operation, in petagrams per decade. Your expression may involve one or more integrals.

Answer: $\frac{1}{20}\int_0^{20} \big(A(t) - S(t)\big)\,dt$ petagrams per decade.

e. [2 points] The San-Ti want to start the first colony on the planet as soon as there are at least 150 Pg of oxygen in the planet's atmosphere, but not before. Assuming there were 100 Pg of oxygen in the planet's atmosphere when the terraforming began, about how many decades must they wait before setting up their first colony? Choose the one best answer below.

7   12   14   17   20   [MORE THAN 20]

4. [11 points] The following parts are unrelated.

a. [3 points] Which of the following limits are equal to 0? Circle **all** correct answers.

i. $\lim_{x \to 0} \frac{x^3 - 4x + 7}{x^4 + 2x}$

ii. $\boxed{\lim_{x \to 0} \frac{x^4 + 2x}{x^3 - 4x + 7}}$

iii. $\boxed{\lim_{x \to \infty} \frac{x^2}{e^x}}$

iv. $\lim_{x \to -\infty} \frac{x^2}{e^x}$

v. $\boxed{\lim_{x \to 0} |x|}$

vi. NONE

b. [2 points] A dose of a drug is injected into a patient's body. The quantity of the drug remaining in the patient's body decays exponentially at a continuous rate of 5% per hour. Which of the following functions could represent the percentage of the original dose which is still remaining in the patient's body after t hours? Circle **the one best** answer.

i. $100e^{0.05t}$

ii. $100e^{1-0.05t}$

iii. $100e^{0.95t}$

iv. $\boxed{100e^{-0.05t}}$

v. $100(1 - e^{0.05t})$

vi. NONE

c. [3 points] The linear approximation to the function $P(x)$ at $x = 1$ is given by $L(x) = e(x-1)+\frac{1}{2}$. Which of the following could be a formula for $P(x)$? Circle **all** correct answers.

i. $\boxed{P(x) = e(x-1) + \frac{1}{2}}$

ii. $P(x) = \frac{1}{2} + e^x$

iii. $\boxed{P(x) = e^x + \frac{1}{2} - e}$

iv. $\boxed{P(x) = \sin(e(x-1)) + \frac{1}{2}}$

v. $P(x) = \cos(e(x-1)) - \frac{1}{2}$

vi. NONE

d. [3 points] A company sells their product for \$5 per unit, and their fixed cost of production is \$2000. If their cost function, in dollars, to produce q units is $C(q)$ and their marginal cost function is $MC(q)$, which of these expressions represents the total profit generated from producing 1000 units of their product? Circle **all** correct answers.

i. $\boxed{5000 - C(1000)}$

ii. $5 - MC(1000)$

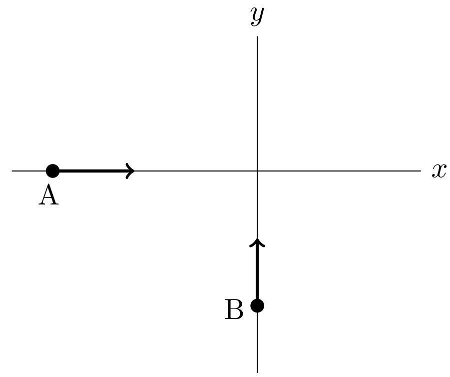
iii. $\boxed{3000 - \int_0^{1000} MC(q)\, dq}$

iv. $2000 + \int_0^{1000} (5 - MC(q))\, dq$

v. $\int_0^{1000} (5q - C(q))\, dq - 2000$

vi. NONE

5. [11 points]
Two cars, A and B, are driving along straight roads towards a common intersection, as pictured to the right. Car A is driving due east, and Car B due north. At 12pm, Car A is west of the intersection and Car B south of it, so that they will both reach the intersection sometime later that afternoon.



Let $u(t)$ be the instantaneous velocity of Car A t hours after 12pm, and $v(t)$ the instantaneous velocity of Car B t hours after 12pm, both in kilometers per hour (kph). Assume $u(t)$ is positive when Car A is traveling east, and $v(t)$ is positive when Car B is traveling north.

a. [2 points] Assume Car A reaches the intersection a hours after 12pm, and Car B reaches the intersection b hours after 12pm. Circle the **one best** practical interpretation of the equation

$$\int_0^a u(t)\,dt = \int_0^b v(t)\,dt.$$

i. The two cars arrive at the intersection at the same time.

ii. The two cars pass through the intersection traveling the same speed.

iii. The two cars start the same distance away from the intersection at 12pm.

iv. The difference between the two cars' initial speeds at 12pm equals the difference between their speeds at the moments when they pass through the intersection.

v. NONE OF THESE.

b. [3 points] Circle the **one** equation below that **best** represents the following statement: "Car A is traveling about 5 kph slower at 2:03pm than it was traveling at 2pm."

(i) $(u^{-1})'(5) = \frac{3}{60}$ (iii) $u'(2.05) = -100$ (v) $\int_2^{2.03} u(t)\,dt = -5/3$

(ii) $(u^{-1})'(5) = 2$ (iv) $u'(2) = -5/3$ (vi) $\int_2^{2.03} u'(t)\,dt = -5/3$

c. [3 points] Write a sentence that gives a practical interpretation of the equation

$$\int_0^3 u(t)\,dt = 296.$$

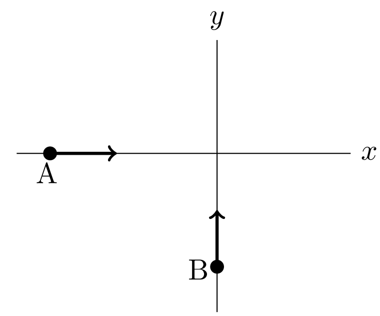
Solution: At 3pm, Car A's position is 296km to the east of its position at 12pm.

d. [3 points] Write a mathematical equation that represents the following statement: "The average velocity of Car B over the three hours between 12pm and 3pm is 99 kph."

Answer: $\frac{1}{3}\int_0^3 v(t)\,dt = 99$

6. [6 points]

Assume the same setup in this problem as in the previous problem, and now additionally assume that at 1pm, Car A is 8 km west of the intersection traveling east at 50 kph, while Car B is 6 km south of the intersection traveling north at 100 kph. How fast is the distance between the two cars changing at 1pm?



Solution: Let us write x for the x-coordinate of Car A's location along its road t hours after 12pm, y for the y-coordinate of Car B's location along its road t hours after 12pm, and z for the distance between Cars A and B t hours after 12pm. So $x^2 + y^2 = z^2$ by the Pythagorean Theorem, and we are looking for $\frac{dz}{dt}$ when $t = 1$. Differentiating $x^2 + y^2 = z^2$ with respect to t gives us

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}. \qquad (1)$$

To find $\frac{dz}{dt}$ when $t = 1$, we plug in $x = -8$, $\frac{dx}{dt} = 50$, $y = -6$, $\frac{dy}{dt} = 100$, and

$$z = \sqrt{(-8)^2 + (-6)^2} = \sqrt{100} = 10$$

into Equation (1) and solve for $\frac{dz}{dt}$ to get:

$$\frac{dz}{dt} = \frac{2(-8)(50) + 2(-6)(100)}{2 \cdot 10} = -8 \cdot 5 - 6 \cdot 10 = -40 - 60 = -100.$$

This means the distance between the two cars is *decreasing* at a rate of 100 kph at 1pm.

Answer: The distance is INCREASING [DECREASING] at a rate of 100 kph.

7. [6 points] Find all local extrema of the function $p(x) = x^5 - 5x^4 + 5x^3 + 1$, and classify each as a local maximum or a local minimum. If there are none of a particular type, write NONE. Use calculus to find your answers, and make sure you **show enough evidence** to justify your conclusions.

Solution: We need to find the critical points of $p(x)$ and test them to determine which ones are local extrema. Differentiating $p(x)$ and factoring, we get

$$p'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x-3)(x-1).$$

The polynomial function $p(x)$ is differentiable everywhere, so its only critical points are the roots of $p'(x)$, which are $x = 0, 1, 3$. Now we use sign logic to find the sign of $p'(x) = 5x^2(x-3)(x-1)$ in each interval determined by these critical points:

$p'(x)$:	$+$		$+$		$-$		$+$
	$+--$	0	$+--$	1	$+-+$	3	$+++$

Alternatively, we could find these signs by plugging test points from each interval into $p'(x)$, or by sketching a graph of $p'(x)$. However we determine them, it follows from the First Derivative Test that $p(x)$ has a local min at $x = 3$ and a local max at $x = 1$, while $x = 0$ is not a local extremum of $p(x)$ since $p'(x)$ does not change sign at $x = 0$.

Answer: Local min(s) at $x =$ 3

Answer: Local max(es) at $x =$ 1

8. [8 points] After its first month of production, Alana's company has manufactured 1.2 million Stan Lee cups. Now she is trying to decide whether it would be in the company's interest to halt production here, or else continue manufacturing more cups.

Suppose $C(q)$ and $R(q)$ are the cost and revenue functions, respectively, of the company producing and selling q million cups, and let $\pi(q) = R(q) - C(q)$ be the profit function. Assume $C(q)$ and $R(q)$ are differentiable for all $q > 0$, and that Alana can sell every cup she produces.

The parts below describe different scenarios that are independent of each other. In each scenario, determine whether Alana should *continue producing more cups* or instead *halt production at 1.2 million cups* in order to maximize profit. If there is not enough information to decide, answer NEI. Circle the one best answer.

a. [1 point] Suppose $MR(1.2) > MC(1.2)$.

[PRODUCE MORE CUPS] HALT PRODUCTION AT 1.2 MILLION NEI

b. [1 point] Suppose $C(1.2) > R(1.2)$.

PRODUCE MORE CUPS HALT PRODUCTION AT 1.2 MILLION [NEI]

c. [2 points] Suppose $\pi'(1.2) = 0$.

PRODUCE MORE CUPS HALT PRODUCTION AT 1.2 MILLION [NEI]

d. [2 points] Suppose $\pi'(q) < 0$ for all $q > 1.2$.

PRODUCE MORE CUPS [HALT PRODUCTION AT 1.2 MILLION] NEI

e. [2 points] Suppose $\pi(q)$ has a local minimum at $q = 1.2$.

[PRODUCE MORE CUPS] HALT PRODUCTION AT 1.2 MILLION NEI

9. [11 points] Consider the family of ellipses $\mathcal{C}$ defined implicitly by the equation

$$k^{-4}x^2 + e^{2k}y^2 = 1,$$

where $k > 0$ is a parameter.

a. [2 points] Find the unique value of k such that the point $(0, \frac{1}{2})$ lies on the ellipse $\mathcal{C}$.

Solution: We plug $x = 0$ and $y = \frac{1}{2}$ into the equation that defines $\mathcal{C}$, and solve for k:

$$\begin{aligned} k^{-4} \cdot 0^2 + e^{2k}(1/2)^2 &= 1, \\ e^{2k} &= 4, \\ k &= \frac{1}{2}\ln 4 = \ln 2. \end{aligned}$$

Answer: $k =$ $\ln 2$

This problem continues on the next page

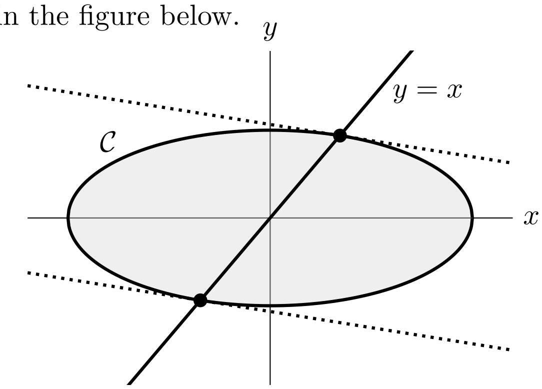
Recall from the previous page that $\mathcal{C}$ is the family of ellipses defined implicitly by the equation

$$k^{-4}x^2 + e^{2k}y^2 = 1,$$

where $k > 0$ is a parameter. The curve $\mathcal{C}$ is pictured for $k = 1$ in the figure below.

b. [4 points]

For every $k > 0$, the ellipse $\mathcal{C}$ intersects the line $y = x$ in two points, as shown in the figure to the right for $k = 1$. The lines tangent to $\mathcal{C}$ at these two points are parallel to each other. Find the *slope* of these dotted lines, in terms of the parameter k.



Solution: The slope of the dotted lines will be the value of $\frac{dy}{dx}$ at the two points given in the picture. So we implicitly differentiate the equation that defines $\mathcal{C}$ with respect to x:

$$2k^{-4}x + 2e^{2k}y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx} = \frac{-2k^{-4}x}{2e^{2k}y} = \frac{-1}{k^4e^{2k}} \cdot \frac{x}{y}.$$

Since $x = y$ at the two points in question, the term $\frac{x}{y}$ in our expression above reduces to 1, and we obtain a slope of $\frac{dy}{dx} = \frac{-1}{k^4e^{2k}} = -k^{-4}e^{-2k}$.

Answer: slope = $-k^{-4}e^{-2k}$

c. [5 points] The area A of the elliptical region bounded by $\mathcal{C}$ is given in terms of k by

$$A = \pi k^2 e^{-k}.$$

Find the value of k (with $k > 0$) that makes the area of this region as large as possible. Use calculus to find your answer, and show enough evidence to justify your conclusions.

Solution: We need to maximize the function $A = f(k) = \pi k^2 e^{-k}$ on the domain $k > 0$. Taking a derivative, we get

$$\frac{dA}{dk} = 2\pi k e^{-k} - \pi k^2 e^{-k} = \pi k e^{-k}(2 - k).$$

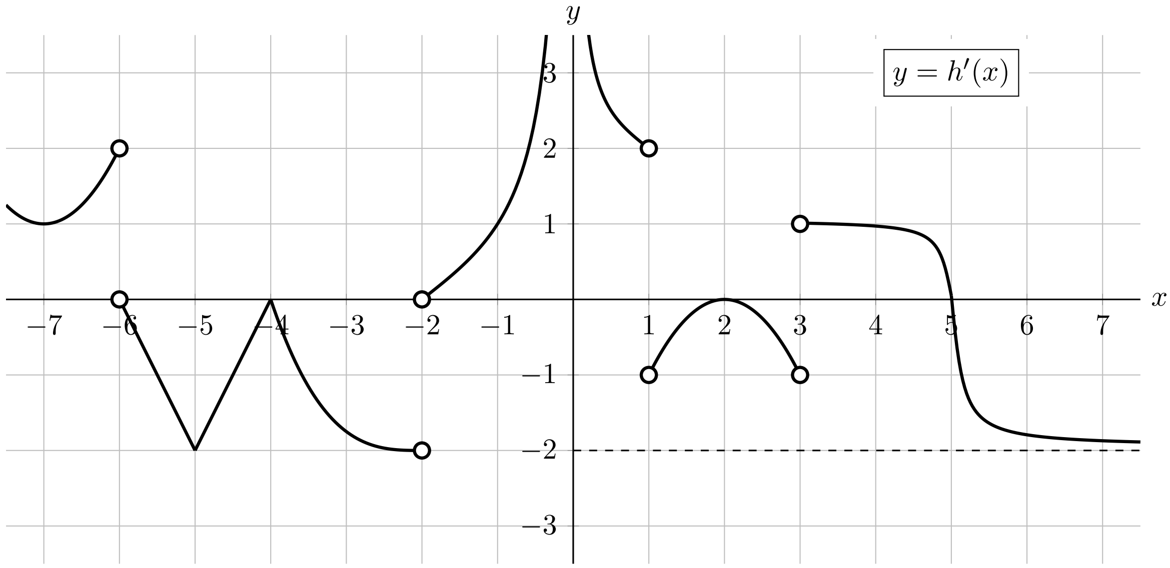
This gives us critical points of $k = 0$ (which we may ignore, since it lies outside our domain) and $k = 2$. Note that $f(2) = \frac{4\pi}{e^2} > 0$. Considering the endpoints of our domain, we have

$$\lim_{k\to 0^+} \frac{dA}{dk} = 0 = \lim_{k\to\infty} \frac{dA}{dk},$$

so $k = 2$ must be the global maximum of $f(k)$ on the interval $(0, \infty)$.

Answer: k = 2

10. [14 points] A function $h(x)$ is defined and continuous on $(-\infty, \infty)$. A portion of the graph of $h'(x)$, **the derivative of** $h(x)$, is shown below. Note that $x = 0$ is a vertical asymptote of $y = h'(x)$ and that $y = -2$ is a horizontal asymptote, as indicated.



In each part **a.–f.** below, circle all correct choices.

a. [2 points] At which of the following value(s) does $h(x)$ have a critical point?

$x = -7$ $x = -5$ $\boxed{x = -2}$ $\boxed{x = 0}$ $\boxed{x = 5}$ NONE OF THESE

b. [2 points] At which of the following value(s) does $h(x)$ have a local maximum?

$\boxed{x = -6}$ $x = -4$ $x = 0$ $\boxed{x = 1}$ $\boxed{x = 5}$ NONE OF THESE

c. [2 points] At which of the following value(s) does $h'(x)$ have a local maximum?

$x = -7$ $\boxed{x = -4}$ $x = 0$ $\boxed{x = 2}$ $x = 5$ NONE OF THESE

d. [2 points] At which of the following value(s) does $h(x)$ have an inflection point?

$\boxed{x = -6}$ $\boxed{x = 0}$ $\boxed{x = 1}$ $\boxed{x = 2}$ $x = 5$ NONE OF THESE

e. [2 points] On which of the following intervals does $h'(x)$ satisfy the hypotheses of the Mean Value Theorem?

$[-5.5, -4.5]$ $\boxed{[-4, -3]}$ $[-4, 5]$ $\boxed{[4, 5]}$ NONE OF THESE

f. [2 points] On which of the following intervals does $h'(x)$ satisfy the conclusion of the Mean Value Theorem?

$[-5.5, -4.5]$ $\boxed{[-4, -3]}$ $\boxed{[-4, 5]}$ $\boxed{[4, 5]}$ NONE OF THESE

g. [2 points] Find the following limits. If there is not enough information, write NEI. If a limit diverges to ∞ or $-\infty$ or if the limit does not exist for any other reason, write DNE.

$\lim_{x \to \infty} h(x) = \boxed{\text{DNE } (-\infty)}$ $\lim_{x \to \infty} h'(x) = \boxed{-2}$